# Analysis of a Class of Strange Attractors

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The notion of a “strange attractor” has been common knowledge in dynamical systems for more than two decades and has captured the attention of scientists in other disciplines. Rigorous mathematical analysis, however, has not kept pace with these developments. Among the examples that have been studied are the Lorenz attractors ([G], [Ro], [Ry], [W2]) and the Hénon maps ([BC2], [BY1], [BY2]). In both of these examples, the attractors are closely related to certain 1-dimensional maps.

This paper is a general study of attractors that are derived, in some fashion, from 1-dimensional maps. The unstable manifolds of the resulting attractors are 1-dimensional; the attractors themselves live in dimensions $\geq 2$. We will limit ourselves to *discrete time* and *smooth* maps; thus our study includes as a special case the Hénon attractors but not the Lorenz flows. Our investigation proceeds in several different directions, ranging from local analysis to global geometry to statistical properties. Although the attractors in our class have a number of features in common with Axiom A attractors and with piecewise monotonic maps in 1-dimension, the reader will find that these two theories together are far from adequate for handling the new complexities that arise.

We now give a general description of the broad category of attractors that are the objects of our study. Our results apply to a subset of this class. Let $f : N \to N$ be a self-map of a circle or an interval, and let $M = N \times D_n$ where $D_n$ is an $n$-dimensional disk. Identifying $N$ with $N \times \{0\} \subset M$, we perturb $f$ into an embedding of $N$ into $M$, and then extend it to an embedding $T$ of $M$ into itself. The attractor of interest to us is given by $\Omega = \cap_{i \geq 0} T^i M$. If $f(z) = z^2$ and $M$ is the solid torus $S^1 \times D_2$, then $\Omega$ is the well known solenoid. We propose to replace the map $f(z) = z^2$ in the standard solenoid by an arbitrary smooth map. For lack of a better name, let us call these **(generalized) solenoidal attractors**.

In addition to the two examples we have encountered, namely the Axiom A solenoid [Sm] (with $f(z) = z^2$) and the Hénon maps [H] (with $f_a(x) = 1 - ax^2,\ x \in [-1, 1]$), other known examples of solenoidal attractors include dissipative twist maps [Bi], the most standard of which can be realized as a suitable perturbation of $f(x) = x + \frac{K}{2\pi}\sin(2\pi x)$, $x \in \mathbb{R}/\mathbb{Z}$, and certain periodically forced nonlinear oscillators ([Lev]; see also [GH]).

Since the picture is well understood when $f$ is uniformly expanding, we are primarily interested in the case where $f$ has critical points. When the critical orbits of $f$ tend to attractive cycles, the dynamics of the $T$ (assuming the perturbations are small) is also quite simple: the stable periodic orbits persist, and the complement of their basins consists of “horseshoes” and their stable manifolds. We focus, therefore, on 1-dimensional maps $f$ that are “chaotic” with no stable equilibria. We mention two important differences between this situation and $f(z) = z^2$. First, $T|\Omega$ in general cannot be realized as the inverse limit of $f$; it is more complicated. Second, while $f(z) = z^2$ gives rise to essentially one attractor – in the sense that two different perturbations $T$ and $T'$ can be conjugated by a homeomorphism $C^0$-near the identity – an arbitrary $f$ can (and does) give rise to infinitely many “different” attractors.

We now give a more precise description of the setting to which our results apply.

## Setting of this paper

Our results are for attractors that arise from perturbations of circle or interval maps. For definiteness, we assume $N = S^1$, $M$ is an annulus, and impose the following conditions on $T$ to make the dynamics more tractable. To ensure that $T$ is predominantly hyperbolic, it is necessary to start with a 1-dimensional map with sufficiently strong expanding properties. We assume $f$ satisfies the Misiurewicz condition, i.e. $f$ is an arbitrary piecewise monotonic map with the property that its forward critical orbits stay away from its critical points. We consider a 2-parameter family $\{T_{a,b}\}$ through $f$, using the parameter $a$ to control movements along the circle and $b$ to "unfold" the 1-dimensional maps in the second direction. Mild transversality conditions are assumed on the 2-parameter family, and the maps $T_{a,b}$ are required to be diffeomorphisms for $b > 0$.

This paper concerns the parameter range where $b$ is small, that is, where $T_{a,b}$ is **strongly dissipative**. Detailed studies are presented for maps corresponding to a positive measure set of parameters in this range.

We mention some small generalizations. When $N = S^1$ and $|\deg(f)| > 1$, at least three dimensions are needed for $T_{a,b}$ to be globally injective. An extension of our techniques gives essentially the same results; details will appear elsewhere. Another possible extension, which we will not discuss, is to replace $S^1$ with branched-1-manifolds (see [W1]).

## Overview of Results

**Selection of parameters and the critical set.** Given $\{T_{a,b}\}$, the goal of this step is to select a positive measure set of "good" parameters corresponding to maps that one can control in certain ways. Our criteria for parameter selection are similar to those of Benedicks and Carleson [BC2], which in turn draws its inspiration from previous work on 1-dimensional maps, from [BC1] and [CE] in particular. In this approach, one inductively identifies and controls an object called the *critical set*, which one hopes will play the role of critical points in 1-dimension. Our inductive process gives new information not available in [BC2]. We obtain a systematic description of the structure of the map near the critical set, which we realize in a Cantor construction as the intersection of a nested sequence of sets each one of which is a union of rectangles with known geometric properties. This detailed knowledge of the critical set is crucial in many of our results. Another departure from [BC2] is that our analysis is based on simple geometric conditions, whereas the equations of the Hénon maps are used in many computations there. We also give a more complete treatment of parameter-space issues than in previously published works.

*The results below hold for maps corresponding to the parameters selected.*

**Hyperbolic behavior.** We prove that compact invariant sets disjoint from the critical set are uniformly hyperbolic, with hyperbolicity getting weaker as one approaches the critical set. Our analysis also gives information on the nonuniform character of hyperbolicity in the basin.

**Statistical properties.** We construct Sinai-Ruelle-Bowen (SRB) measures on our attractors, bound the number of ergodic SRB measures by the number of critical points of the generating 1-dimensional map, and show that with respect to Lebesgue measure, almost every point in the basin is generic with respect to an ergodic SRB measure. Appealing to the abstract results in [Y3] and [Y4], we prove that the attractors in our class enjoy a Central Limit Theorem and have exponential decay of correlations on their mixing components. The corresponding results for Axiom A attractors have been known since the 1970s ([S2], [R1], [R2]). For the Hénon family near $a = 2,\ b = 0$, SRB measures and their statistical properties were studied in [BY1] and [BY2], and the basin property in [BV].

**Global geometry.** The approximate shape and complexity of the Axiom A solenoid is given by a small tubular neighborhood of a simple closed curve winding around the solid torus $2^k$ times. In analogy with piecewise monotonic maps in 1-dimension, we introduce the notion of "monotone branches" and show that our attractors have arbitrarily fine neighborhoods that are unions of finitely many of these branches. From the way these branches fit together one obtains a certain insight into the differences between one and two-dimensional maps.

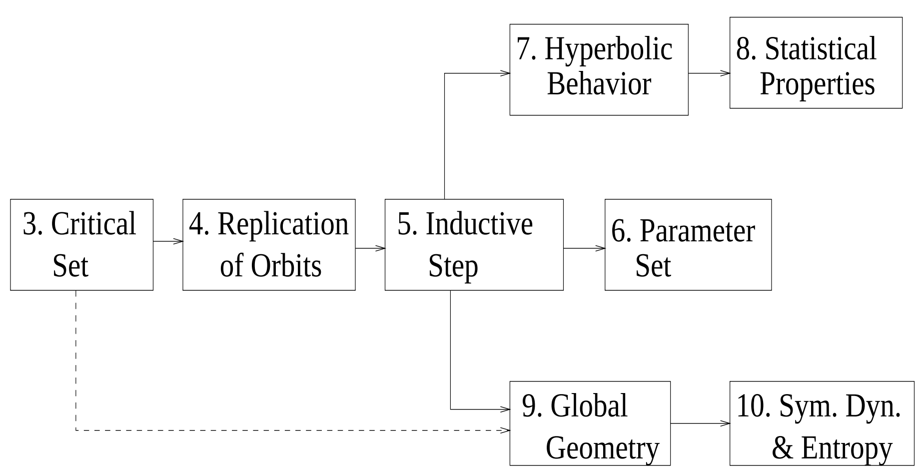
**Symbolic dynamics and topological entropy.** The geometric considerations above make it possible to code unambiguously all orbits on the attractor, representing the dynamics of the map by a shift operator acting on symbol sequences generated by a finite alphabet. In the non-Axiom A case this shift is not of finite type. The coding we give is an honest reflection of the true locations visited by an orbit relative to the "components" of the critical set. Kneading sequences for critical orbits are well defined, and monotone branches correspond to cylinder sets. This symbolic representation is nearly one-to-one, allowing us to deduce from it the existence of equilibrium states and various formulas for computing topological entropy. For results in this direction for Axiom A attractors, see [Bo] and the references therein.

It is our hope that by formulating simple, checkable conditions as we have done in Sect. 1.1, one can determine readily if the results of this paper apply to a given situation. We illustrate this for generic **homoclinic bifurcations** of 2-dimensional diffeomorphisms, recovering the result in [MV] (which extends [BC2] to this setting) and obtaining immediately for the attractors in question the dynamical picture above.

The results of this paper open the door to a host of questions for the class of attractors being studied. For example, with the information available, extensions of the theory of equilibrium states to the present setting may be possible (see e.g. [Bo], [R2], [K]). Our results on symbolic dynamics lead naturally to questions on the zeta function (see e.g. [Bal], [PP], [R3]). With kneading sequences for critical orbits being well defined, it is reasonable to consider the possibility of a kneading theory (see [C], [MT]). In a different direction, the notion of monotone branches leads to questions about global topological structures and prime ends (e.g. [Bar]).

We mention some other related works, omitting specific references to 1-dimensional dynamics (see the reference in [dMvS]). For results on piecewise uniformly hyperbolic attractors, see e.g. [CL], [I1], [I2], [M2] and [Y1]. For the statistical properties of billiards, see e.g. [S1], [BSC1], [BSC2] and [Y3]. Closer to the setting of this paper are [DRV] and [MV], in which Viana *et. al.* extended the analysis in [BC2] to "Hénon-like" maps and found applications for these extensions. See also [V]. Jakobson and Newhouse have announced that they have reproduced, using different methods, the results in [BC2] and [BY1]. We have been told that Luzzatto has done work in this direction, and that Palis and Yoccoz have results for certain non-attracting sets that arise in homoclinic bifurcations.

This paper is by and large self-contained — with the exception of Section 6, where two results from 1-dimensional maps are quoted without proof, and Section 8, where previous work of the second-named author is used. Proofs that are computational in nature have been put in the Appendix so that they will not obstruct the main flow of ideas. In a paper as long as this one, it might be useful to indicate the logical connections among the various sections. After Section 1, we recommend at least looking through Section 2, in which we introduce much of the basic vocabulary for subsequent sections. The other sections are connected as indicated. (For example, the technical content of Section 6 is not needed for reading Sections 7-10.)



# 1 Statements of Results

## 1.1 Setting

For definiteness, Theorems 1–7 are stated in the context of attractors that arise from perturbations of *circle* maps. For the interval case, see Sect. 1.5.

Let $A = S^1 \times [-1, 1]$. We consider 2-parameter families of maps $\{T_{a,b}\}$ where for each $(a, b)$, $T_{a,b} : A \to A$ is a self-map of $A$ and $(x, y, a, b) \mapsto T_{a,b}(x, y)$ is $C^3$. The class of 2-parameter families $\{T_{a,b}\}$ to which our results apply are constructed via the following four steps.

**Step I.** Let $f : S^1 \to S^1$ satisfy the following **Misiurewicz conditions**, i.e. letting $C = \{x : f'(x) = 0\}$, we assume:

1. $f''(x) \neq 0$ for all $x \in C$;
2. $f$ has negative Schwarzian derivative on $S^1 \setminus C$;
3. there is no $x \in S^1$ with $f^n(x) = x$ and $|(f^n)'(x)| \leq 1$;
4. for all $x \in C$, $\inf_{n>0} d(f^n x, C) > 0$.

Observe that for $p \in S^1$ with $\inf_{n \geq 0} d(f^n p, C) > 0$, if $g$ is sufficiently near $f$ in the $C^2$ sense, then there is a unique point $p(g)$ having the same symbolic dynamics with respect to $g$ as $p$ does with respect to $f$. If $\{f_a\}$ is a 1-parameter family through $f$, then for those $a$ for which it makes sense, we will call $p(a) = p(f_a)$ the *continuation* of $p$. For $x \in C$, we let $x(a)$ denote the corresponding critical point of $f_a$.

**Step II.** Let $f$ be as in Step I, and let $\{f_a\}$, $a \in [a_0, a_1]$, be a 1-parameter family of maps from $S^1$ to $S^1$ with $f = f_{a^*}$ for some $a^* \in [a_0, a_1]$. We require that $\{f_a\}$ satisfy the following **transversality condition**[3]: For every $x \in C$, if $p = f(x)$, then

$$\frac{d}{da} f_a(x(a)) \neq \frac{d}{da} p(a) \qquad \text{at} \quad a = a^*. \tag{1}$$

**Step III.** Let $\{f_a\}$ be as in Step II. Identifying $S^1$ with $S^1 \times \{0\} \subset A$, we extend $\{f_a\}$ to a 2-parameter family $\{f_{a,b}\}$, $a \in [a_0, a_1]$, $b \in [0, b_1]$, where $f_{a,b} : S^1 \to A$ is such that $f_{a,0} = f_a$ and $f_{a,b}$ is an *embedding* for $b > 0$.

**Step IV.** Finally, we extend $f_{a,b}$ to $T_{a,b} : A \to A$ in such a way that $T_{a,0}(A) \subset S^1 \times \{0\}$ and for $b > 0$, $T_{a,b}$ maps $A$ diffeomorphically onto its image. We further impose the following **non-degeneracy condition**[4] on the map $T_{a^*,0}$:

$$\partial_y T_{a^*,0}(x, 0) \neq 0 \quad \text{whenever} \quad f'_{a^*}(x) = 0. \tag{2}$$

[3]This transversality condition is used in [TTY].

[4]This condition is not assumed in [MV] or [BV]. Their regularity condition on $|\det(DT)|$ and bound on the perturbation term, however, imply a condition which is similar (though not equivalent) to (2) and which serves a similar purpose.

This completes our construction of admissible families $\{T_{a,b}\}$. We remark that the transversality and non-degeneracy conditions in Steps II and IV are generic. Thinking in terms of normal neighborhoods, one constructs easily for a given $f_{a,b}$ extensions of the type in Step IV; the signs of the $\partial y$-derivatives at the critical points of $f_{a^*}$ are determined by the orientations of the turns of $f_{a,b}$ at the corresponding points. Step III is feasible if and only if the degree of $f$ is $0, 1$ or $-1$. If $|\deg(f)| > 1$, an extra dimension is needed; this will be treated in a separate paper.

Observe that for $b > 0$, $T_{a,b}$ has the general form

$$T_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} F(x,y,a) + b\ u(x,y,a,b) \\ b\ v(x,y,a,b) \end{pmatrix} \tag{3}$$

where $F(x,y,a) = T^1_{a,0}(x,y)$, the first component of $T_{a,0}(x,y)$, and the $C^2$ norms of $(x,y,a) \mapsto u(x,y,a,b)$ and $v(x,y,a,b)$ are uniformly bounded for all $b \in (0, b_1]$.[5]

In terms of differentiability assumptions, the following slightly more technical formulation corresponds exactly to what is used:

(i) $(x,y,a) \mapsto T_{a,b}(x,y)$ has uniformly bounded $C^3$-norms in $b$,

(ii) $T_{a,b}$ is of the form (3) with uniformly bounded $C^2$ norms for $u$ and $v$.

**Notation.** Given $\{T_{a,b}\}$, constants that are determined entirely by the family $\{T_{a,b}\}$ will be referred to as **system constants**. Except where declared otherwise, the letter $K$ is reserved throughout this article for use as a **generic system constant**, meaning a system constant that is allowed to change from statement to statement (the other system constants are fixed). We will use $K_1, K_2$ etc. where $K$ appears in more than one role in the same statement.

Let $K$ be such that $T_{a,b}(A) \subset R_0 := S^1 \times [-Kb, Kb]$ for all $(a,b)$. It is convenient for us to work with $R_0$ instead of $A$. For $T = T_{a,b}$, let $R_n = T^n R_0$. Then $\{R_n\}$ is a decreasing sequence of neighborhoods of the **attractor** $\Omega := \cap_{n=0}^{\infty} R_n = \cap_{n=0}^{\infty} T^n A$.

## 1.2 Critical set and hyperbolic behavior

Our first theorem identifies, for each map $T$ corresponding to a selected set of parameters, a fractal set $\mathcal{C}$ chosen to play the role of the critical set in 1-dimension. This set will be called the **critical set** of $T$. Our parameter selection imposes strong hyperbolic properties on orbits starting from $\mathcal{C}$ in the hope that these properties will be passed on to the rest of the system. The geometric structure near $\mathcal{C}$, which is described in some detail in Theorem 1, is crucial for many of our later results.

For $z_0 \in R_0$, let $z_i = T^i z_0$. If $w_0$ is a tangent vector at $z_0$, let $w_i = DT^i(z_0)w_0$. A curve in $R_0$ is called a $C^2(b)$**-curve** if the slopes of its tangent vectors are $\mathcal{O}(b)$ and its curvature is everywhere $\mathcal{O}(b)$.

[5]This is a calculus exercise: Observe that $bu$ extends to a $C^3$ function $g$ on $\{b \geq 0\}$ with $g|\{b = 0\} = 0$. Writing $\partial^2 = \frac{\partial^2}{\partial z_1 \partial z_2}$ where $z_i = x, y$ or $a$, we then check that $\partial^2 u$ extends to a continuous function $h$ on $\{b \geq 0\}$ with $h = \frac{\partial}{\partial b}\partial^2 g$ on $\{b = 0\}$.

**Theorem 1 (Parameter selection and the critical set)** *Given $\{T_{a,b}\}$ as in Sect. 1.1, there is a positive measure set $\Delta \subset [a_0, a_1] \times (0, b_1]$ such that (1) and (2) below hold for $T = T_{a,b}$ for all $(a, b) \in \Delta$. The set $\Delta$ is located near $a = a^*$ and $b = 0$; it has the property that for all sufficiently small $b$, $\Delta_b := \{a : (a, b) \in \Delta\}$ has positive 1-dimensional Lebesgue measure. The constants $\alpha, \delta, c > 0$ and $0 < \rho < 1$ below are system constants, and $b << \alpha, \delta, \rho, e^{-c}$ for all $(a, b) \in \Delta$.*

*(1)* **Critical regions and critical set.** *There is a Cantor set $\mathcal{C} \subset \Omega$ called the critical set given by $\mathcal{C} = \cap_{k=0}^{\infty} \mathcal{C}^{(k)}$ where the $\mathcal{C}^{(k)}$ are a decreasing sequence of neighborhoods of $\mathcal{C}$ called critical regions.*

**Geometrically***:*

*(i) $\mathcal{C}^{(0)} = \{(x, y) \in R_0 : d(x, C) < \delta\}$ where $C$ is the set of critical points of $f$.*

*(ii) $\mathcal{C}^{(k)}$ has a finite number of components called $Q^{(k)}$ each one of which is diffeomorphic to a rectangle. The boundary of $Q^{(k)}$ is made up of two $C^2(b)$ segments of $\partial R_k$ connected by two vertical lines: the horizontal boundaries are $\approx \min(2\delta, \rho^k)$ in length, and the Hausdorff distance between them is $\mathcal{O}(b^{\frac{k}{2}})$.*

*(iii) $\mathcal{C}^{(k)}$ is related to $\mathcal{C}^{(k-1)}$ as follows: $Q^{(k-1)} \cap R_k$ has at most finitely many components, each one of which lies between two $C^2(b)$ subsegments of $\partial R_k$ that stretch across $Q^{(k-1)}$ as shown. Each component of $Q^{(k-1)} \cap R_k$ contains exactly one component of $\mathcal{C}^{(k)}$.*

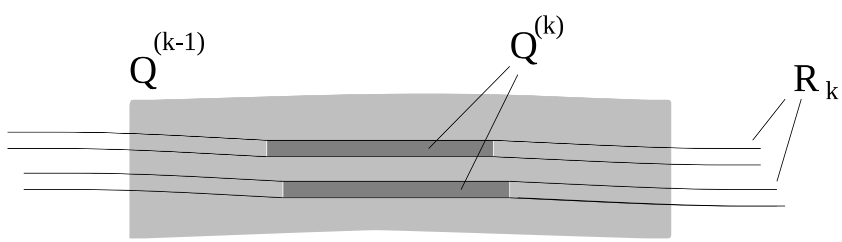



Figure 1 Critical regions

**Dynamically***: On each horizontal boundary $\gamma$ of $Q^{(k)}$ there is a unique point $z$ located within $\mathcal{O}(b^{\frac{k}{4}})$ of the midpoint of $\gamma$ with the property that if $\tau$ is the unit tangent vector to $\gamma$ at $z$, then $DT^n(z)\tau$ decreases in length exponentially as $n$ tends to $\infty$.*

*(2)* **Properties of critical orbits.** *For $z \in R_0$, let $d_{\mathcal{C}}(z)$ denote the following notion of "distance to the critical set": If $z \notin \mathcal{C}^{(0)}$, let $d_{\mathcal{C}}(z) = \delta$; if $z \in \mathcal{C}^{(0)} \setminus \mathcal{C}$, let $k$ be the largest number with $z \in \mathcal{C}^{(k)}$, and define $d_{\mathcal{C}}(z)$ to be the horizontal distance between $z$ and the midpoint of the component of $\mathcal{C}^{(k)}$ containing $z$. Then for all $z_0 \in \mathcal{C}$:*

*(i)* $d_{\mathcal{C}}(z_j) \geq e^{-\alpha j}$ *for all* $j > 0$*;*

*(ii)* $\|DT^j(z_0)\binom{0}{1}\| \geq K^{-1}e^{cj}$ *for all* $j > 0$.

**Theorems 2-7 apply to $T = T_{a,b}$, $(a, b) \in \Delta$, where $\Delta$ is as in Theorem 1.**

Our next theorem is about the abundance of hyperbolic behavior on the attractor and in the basin. A compact $T$-invariant set $\Lambda$ is called **uniformly hyperbolic** if there is a splitting of the tangent bundle over $\Lambda$ into invariant subbundles $E^u \oplus E^s$ such that for some $C, \lambda > 1$, we have, for all $n \geq 1$, $\|DT^n v\| \leq C\lambda^{-n}\|v\|$ for all $v \in E^s$ and $\|DT^{-n}v\| \leq C\lambda^{-n}\|v\|$ for all $v \in E^u$.

**Theorem 2 (Hyperbolic behavior)**

*(1) Let*

$$\Omega_\varepsilon := \{z_0 \in \Omega : d_{\mathcal{C}}(z_n) \geq \varepsilon \quad \forall n \in \mathbb{Z}\}.$$

*(i) For every $\varepsilon > 0$, $\Omega_\varepsilon$ is uniformly hyperbolic. In fact, independent of $\varepsilon$, $\lambda$ in the definition of hyperbolicity can be taken to be $\approx e^{\frac{c}{3}}$ where $c$ is as in Theorem 1. In particular, for evevry periodic point $z \in \Omega$ with $T^q z = z$, $\|DT^q|E^u(z)\| \geq K^{-1}e^{\frac{c}{3}q}$.*

*(ii) As $\varepsilon \to 0$, the hyperbolicity on $\Omega_\varepsilon$ deteriorates in the sense that $C \to \infty$ and the minimum angle between $E^u$ and $E^s$ tends to zero.*

*(iii) $\Omega = \overline{\cup_{\varepsilon>0}\Omega_\varepsilon}$ provided a surjective condition of the type (*) below is assumed.*

*(2) Under the regularity conditions (**) below, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DT^n(z_0)\| \geq \frac{c}{3}$$

*for Lebesgue-almost every $z_0 \in R_0$.*

The two technical conditions used in parts (1)(iii) and (2) of Theorem 2 are:

(*) Let $J_1, \cdots J_r$ be the intervals of monotonicity of $f$. Then for each $i$, there exists $j$ such that $f(J_j) \supset J_i$.

(**) There exist $K_1, K_2 > 0$ such that for all $z \in R_0$,

$$K_1^{-1}b \leq |\det(DT_{a,b}(z))| \leq K_2 b$$

We remark that Theorem 2(1) confirms that $\mathcal{C}$ is the sole source of nonhyperbolicity in the system. Part (2) expresses the fact that many orbits experience at least some form of (nonuniform) hyperbolicity. A more detailed discussion is given in Section 7.

## 1.3 SRB Measures and their Statistical Properties

**Definition 1.1** *Let $g : M \to M$ be a diffeomorphism of a manifold. A $g$-invariant Borel probability measure $\mu$ is called an* **SRB measure** *if $g$ has a positive Lyapunov exponents $\mu$-a.e. and the conditional measures of $\mu$ on unstable manifolds are absolutely continuous with respect to the Riemannian measure on these manifolds.*

In the absence of zero Lyapunov exponents, it follows from general hyperbolic theory that an SRB measure has at most a countable number of ergodic components, and that each ergodic component has a positive measure set of generic points. A point $z$ is said to be **generic** with respect to $\mu$ if for every continuous function $\varphi$, $\frac{1}{n}\sum_{i=0}^{n}\varphi(g^i z) \to \int \varphi d\mu$ as $n \to \infty$. See [Led] and [PS].

**Theorem 3 (Existence and ergodic properties of SRB measures)**

*(1) $T$ admits an SRB measure.*

*Assuming condition (**) above, we have the following additional information:*

*(2) $T$ admits at most $r$ ergodic SRB measures $\mu_i$ where $r$ is the cardinality of the critical set of the 1-dimensional map $f$.*

*(3) Lebesgue-a.e. $z_0 \in R_0$ is generic with respect to some $\mu_i$; in fact, Lebesgue-a.e. $z_0 \in R_0$ lies in the stable manifold of a $\mu_i$-typical point in $\Omega$.*

We know from general hyperbolic theory that without zero Lyapunov exponents, ergodic components of SRB measures are, up to finite factors, mixing [Led].

**Theorem 4 (Decay of correlations and Central Limit Thoerem)** *Let $\mu$ be an ergodic SRB measure, which, by taking a power of $T$ if necessary, we assume to be mixing. Then*

*(1) for each $\eta \in (0,1]$, there exists $\lambda = \lambda(\eta) < 1$ such that if $\psi : A \to \mathbb{R}$ is Hölder continuous with exponent $\eta$ and $\varphi \in L^\infty(\mu)$, then there exists $K(\varphi, \psi)$ such that*

$$\left| \int (\varphi \circ T^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| < K(\varphi,\psi)\lambda^n \quad \text{for all } n;$$

*(2) the Central Limit Theorem holds for all Hölder $\varphi$ with $\int \varphi d\mu = 0$, i.e.*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^i \quad \to \quad \mathcal{N}(0,\sigma)$$

*where $\mathcal{N}(0,\sigma)$ is the normal distribution with variance $\sigma^2$; furthermore, $\sigma > 0$ if and only if $\varphi \circ T \neq \psi \circ T - \psi$ for any $\psi$.*

We remark that the word "attractor" has different meanings in the literature (see [Mil] for a discussion). In this article, it is convenient for us to refer to $\Omega$ as "the attractor". Theorem 3 suggests, however, that from a measure-theoretic point of view, it may be more appropriate to regard the supports of the $\mu_i$ as attractors.

## 1.4 Global geometry, symbolic dynamics and topological entropy

A **monotone branch** of $R_n$ is a region diffeomorphic to a rectangle and bordered by two subsegments of $\partial R_n$. Roughly speaking, it is the largest domain of this kind with the property that for $0 \leq i \leq n$, the $x$-coordinates of its $T^{-i}$-image stay inside some interval of monotonicity of $f$, where $f$ is the initial 1-dimensional map from which $\{T_{a,b}\}$ is built. This notion is made precise in Section 9, where a combinatorial tree is introduced to describe the structure of a natural class of monotone branches.

**Theorem 5 (Coarse geometry of attractor)** *There is a sequence of neighborhoods $\tilde{R}_n$ of $\Omega$ with*

$$\tilde{R}_1 \supset \tilde{R}_2 \supset \tilde{R}_3 \supset \cdots \qquad \text{and} \qquad \cap_i \tilde{R}_i = \Omega$$

*such that each $\tilde{R}_n$ is the union of a finite number of monotone branches of $R_k$, $n \leq k \leq n(1 + K\theta)$, where $\theta \sim \frac{-1}{\log b}$.*

Let $\{1, 2, \cdots, k\}$ be a finite alphabet and let $\Sigma_k$ be the set of all bi-infinite sequences $\mathbf{s} = (\cdots, s_{-1}, s_0, s_1, \cdots)$ with $s_i \in \{1, 2, \cdots, k\}$. The shift operator $\sigma : \Sigma_k \to \Sigma_k$ is defined by $(\sigma \mathbf{s})_i = (\mathbf{s})_{i+1}$. For $\Sigma \subset \Sigma_k$, we call $\sigma|\Sigma : \Sigma \to \Sigma$ a **subshift** of the full shift on $k$ symbols if $\Sigma$ is a closed $\sigma$-invariant subset of $\Sigma_k$.

Let $x_1 < x_2 < \cdots < x_r < x_{r+1} = x_1$ be the critical points of $f$. Let $\mathcal{C}_i^{(0)}$ be the component of $\mathcal{C}^{(0)}$ containing $x_i$ and let $\mathcal{C}_i = \mathcal{C} \cap \mathcal{C}_i^{(0)}$. We remark that each $\mathcal{C}_i$ is a fractal set – it is not contained in any smooth curve – and that *a priori* there is no well defined notion of whether a point lies to the left or to the right of $\mathcal{C}_i$.

**Theorem 6 (Coding of orbits on attractor)**

*(1) The critical set $\mathcal{C}$ partitions $\Omega \backslash \mathcal{C}$ into disjoint sets $A_1, A_2, \cdots, A_r$ so that $z \in A_i$ has the interpretation of being "to the right" of $\mathcal{C}_i$ and "to the left" of $\mathcal{C}_{i+1}$.*

*(2) There is a subshift $\sigma : \Sigma \to \Sigma$ of a full shift on finitely many symbols and a continuous surjection $\pi : \Sigma \to \Omega$ such that*

$$T \circ \pi \ = \ \pi \circ \sigma;$$

*$\pi$ is 1-1 except on $\cup_{i=-\infty}^{\infty} T^i \mathcal{C}$, where it is 2-1.*

(3) *Under the additional assumption that* $f[x_j, x_{j+1}] \not\supset S^1$ *for any* $j$, *the coding in (2) is given by (1), i.e. for all* $z_0 \in \Omega \setminus \cup_{i=-\infty}^{\infty} T^i\mathcal{C}$, $\pi^{-1}(z_0)$ *is the unique sequence* $(s_i)_{i=-\infty}^{\infty}$ *with* $z_i \in A_{s_i}$.

**Corollary 1 (Kneading sequences for critical points)** *For every* $z_0 \in \mathcal{C}$, *the itinerary of* $\{z_1, z_2, \cdots\}$ *is uniquely represented by a sequence in* $\Sigma$.

Another consequence of Theorem 6 is the existence of equilibrium states. For a continuous map $g : X \to X$ of a compact metric space and a continuous function $\varphi : X \to \mathbb{R}$, a $g$-invariant Borel probability measure $\mu$ on $X$ is called an **equilibrium state** for $g$ with respect to the potential $\varphi$ if $\mu$ maximizes the quantity

$$\sup \{ \mathrm{h}_\nu(\mathrm{g}) + \int \varphi \mathrm{d}\nu \}$$

where $h_\nu(g)$ denotes the metric entropy of $g$ with respect to $\nu$ and the supremum is taken over all $g$-invariant Borel probability measures $\nu$.

**Corollary 2 (Existence of equilibrium states)** *$T$ has an equilibrium state for every continuous* $\varphi : \Omega \to \mathbb{R}$. *In particular, $T$ admits an invariant Borel probability measure maximizing entropy.*

The **topological entropy** of $g$, written $h_{top}(g)$, is usually defined in terms of open covers of arbitrarily small diameters or in terms of $(n, \varepsilon)$-spanning or separated sets. For precise definitions, see [Wa]. For the class of attractors studied in this paper, $h_{top}(g)$ can be computated in more concrete ways.

In Theorem 6 we saw that every $z_0 \in \Omega$ can be unambiguously associated with one (and occasionally two) symbol sequences in $\Sigma$ determined by the locations of its iterates with respect to the components of the critical set. We will show in Section 10 that in like manner all the points in $R_0$ can be assigned symbol sequences – except that this assignment is not unique. Let us temporarily refer to this as the "fuzzy" coding on $R_0$. Let

$N_n$ = number of distinct $n$-blocks in the coding of $\Omega$;
$\tilde{N}_n$ = number of distinct $n$-blocks in the "fuzzy" coding of $R_0$;
$P_n$ = number of fixed points of $T^n$;
$M_n^{\pm}$ = number of monotone segments in $\partial R_n^{\pm}$, the two boundary components of $R_n$ (see Sect. 9.1 for the precise definition).

**Theorem 7 (Formulas and inequalities for topological entropy)**

*(i)*

$$h_{\rm top}(T) \;=\; \lim_{n\to\infty} \frac{1}{n} \log N_n \;=\; \lim_{n\to\infty} \frac{1}{n} \log \tilde{N}_n \;=\; \lim_{n\to\infty} \frac{1}{n} \log P_n.$$

*(ii)*

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n^{\pm} \;\le\; h_{\rm top}(T) \;\le\; \left( \liminf_{n\to\infty} \frac{1}{n} \log M_n^{\pm} \right) (1 + \frac{K}{\log \frac{1}{b}}).$$

For a 1-dimensional piecewise monotonic map $g$, it is a well known fact that $h_{\rm top}(g)$ is the growth rate of the number of intervals on which $g^n$ is monotonic [MS]. The factor $(1+\frac{K}{\log \frac{1}{b}})$ gives, in a sense, the potential *defect* in measuring the complexity of $T$ via the 1-dimensional curves $\partial R_0^{\pm}$.

## 1.5 Hénon maps and homoclinic bifurcations

Theorems 1–7 are stated for attractors that arise from perturbations of circle maps. We state here, for the record, the corresponding results for *interval maps* and some of their applications. Reduction to the circle case is carried out in Appendix A.1.

**Theorem 8 (Attractors arising from interval maps)** *Let $I$ be a closed interval of finite length, and let $f : I \to I$ be a Misiurewicz map with $f(I) \subset int(I)$. Let $U$ be a neighborhood of $I \times \{0\}$ in $\mathbb{R}^2$, and let $\{T_{a,b}\}$ be a 2-parameter family of maps with $T_{a,b} : U \to \mathbb{R}^2$. We identify $I$ with $I \times \{0\} \subset \mathbb{R}^2$, and assume that $\{T_{a,b}\}$ satisfies the conditions in Steps $II$, $III$ and $IV$ in Sect. 1.1 with $f_{a^*} = f$. Then*

*(i) there exist $K > 0$ and a rectangle $\hat{\Delta} = [a_0, a_1] \times (0, b_1]$ arbitrarily near $(a^*, 0)$ such that for each $(a,b) \in \hat{\Delta}$, $T_{a,b}$ maps $R := I \times [-Kb, Kb]$ strictly into its interior, defining an attractor $\Omega := \bigcap_{n\ge 0} T^n_{a,b} R$;*

*(ii) there is a positive measure set $\Delta \subset \hat{\Delta}$ such that the conclusions of Theorems 1–7 hold for $T = T_{a,b} \mid R$ for all $(a,b) \in \Delta$.*

**Corollary 3 (The Hénon family)** *Let*

$$T_{a,b} : (x,y) \mapsto (1 - ax^2 + y,\ bx), \qquad (x,y) \in \mathbb{R}^2.$$

*Then for every $a^* \in [1.5, 2]$ for which $f_{a^*} : x \mapsto 1 - a^* x^2$ is a Misiurewicz map, the conclusions of Theorem 8 hold. In particular, there is a positive measure set $\Delta$ near $(a^*, 0)$ such that the conclusions of Theorems 1–7 hold for all $T = T_{a,b}$, $(a,b) \in \Delta$. These results are valid for both $b > 0$ and $b < 0$.*

We remark that our method of proof does not distinguish between the orientation preserving and reversing cases of the Hénon maps. When specialized to $a^* = 2$ and $b > 0$, the part of Corollary 3 that corresponds to Theorem 1, part (2), in this paper is a version of the main result of [BC2]. The parts of Corollary 3 that correspond to Theorem 3(1),(2), Theorem 3(3) and Theorem 4 are proved respectively in [BY1], [BV] and [BY2].

Our last result concerns the application of Theorems 1–7 to homoclinic bifurcations. Let $g_\mu$, $\mu \in [0,1]$, be a $C^\infty$ one-parameter family of surface diffeomorphisms unfolding at $\mu = 0$ a nondegenerate tangency of $W^u(p_0)$ and $W^s(p_0)$ where $p_0$ is a hyperbolic fixed point. We assume that the eigenvalues $\lambda$ and $\sigma$ of $Dg_0$ at $p_0$ satisfy $0 < \lambda < 1 < \sigma$ and $\lambda\sigma < 1$, and that they belong in the open and dense set of eigenvalue pairs that meet the hypotheses of Sternberg's linearization theorem. Under these conditions, it is well known (see [PT]) that for all sufficiently large $k$, there is an open set of parameters $\hat{\Delta}_k$ such that for all $\mu \in \hat{\Delta}_k$, $g_\mu$ has a $k$-periodic attractor $\Omega_\mu$ all but finitely many of whose periodic components are located near the fixed point $p_\mu$.

**Theorem 9 (Attractors arising from homoclinic bifurcations)** *Let $g_\mu$ be as above. Then for all sufficiently large $k$, there is a positive measure set of parameters $\Delta_k \subset \hat{\Delta}_k$ for which the following hold: for all $\mu \in \Delta_k$, there is a component $\Omega_\mu^0$ of $\Omega_\mu$ with the property that if $T_\mu$ denotes the restriction of $g_\mu^k$ to a neighborhood of $\Omega_\mu^0$, then the conclusions of Theorems 1–7 hold for $T = T_\mu$.*

Our proof of Theorem 9, which is given in Appendix A.2, consists of observing that the maps $g_\mu^k$ meet the conditions of Theorem 8. The part of Theorem 9 that corresponds to Theorem 1, part (2), in this paper is the main result of [MV].

# 2 Preliminaries

We gather in this section a collection of technical facts used repeatedly in later sections. Most of the proofs are given in Appendix B. Sects. 2.1–2.4 contain material not specific to the family $\{T_{a,b}\}$, and $K$ is not a "system constant" in these subsections.

## 2.1 Linear algebra

Let $M$ be a $2 \times 2$ matrix. Assuming that $M$ is not a scalar multiple of an orthogonal matrix, we say that a unit vector $e$ defines **the most contracted direction** of $M$ if $\|Mu\| \geq \|Me\|$ for all unit vectors $u$. For a sequence of matrices $M_1, M_2, \cdots$, we use $M^{(i)}$ to denote the matrix product $M_i \cdots M_2 M_1$ and $e_i$ to denote the most contracted direction of $M^{(i)}$ when it makes sense.

**Hypotheses for Sect. 2.1** The $M_i$ are $2 \times 2$ matrices; they satisfy $|\det(M_i)| \leq b$ and $\|M_i\| \leq K_0$ where $K_0$ and $b$ are fixed numbers with $K_0 > 1$ and $b << 1$.

**Lemma 2.1** *There exists $K$ depending only on $K_0$ such that if $\|M^{(i)}\| \geq \kappa^i$ and $\|M^{(i-1)}\| \geq \kappa^{i-1}$ for some $\kappa >> \sqrt{b}$, then $e_i$ and $e_{i-1}$ are well-defined, and*

$$\| e_i \times e_{i-1} \| \leq (\frac{Kb}{\kappa^2})^{i-1}.$$

**Corollary 2.1** *If for $1 \leq i \leq n$, $\|M^{(i)}\| \geq \kappa^i$ for some $\kappa >> \sqrt{b}$, then:*

*(a) $\| e_n - e_1 \| < \frac{Kb}{\kappa^2}$;*

*(b) $\|M^{(i)} e_n\| \leq (\frac{Kb}{\kappa^2})^i$ for $1 \leq i \leq n$.*

**Proof:** (a) follows immediately from Lemma 2.1. For (b), since $\|e_n - e_i\| \leq (\frac{Kb}{\kappa^2})^i$, we have $\|M^{(i)} e_n\| \leq \|M^{(i)}(e_n - e_i)\| + \|M^{(i)} e_i\| < K_0^i \cdot (\frac{Kb}{\kappa^2})^i + (\frac{b}{\kappa})^i$. $\square$

Next we consider for each $i$ a 2-parameter family of matrices $M_i(s_1, s_2)$. For the purpose of the next corollary we make the additional assumptions that for $0 < j \leq 2$, $\|\partial^j M_i(s_1, s_2)\| \leq K_0^i$ and $|\partial^j \det(M_i(s_1, s_2))| < K_0^i b$ where $\partial^j$ represents any one of the partial derivatives of order $j$ with respect to $s_1$ or $s_2$. Let $\theta_i(s_1, s_2)$ denote the angle $e_i(s_1, s_2)$ makes with the positive $x$-axis, assuming it makes sense.

**Corollary 2.2** *Suppose that for some $\kappa >> \sqrt{b}$, $\|M^{(i)}(s_1, s_2)\| \geq \kappa^i$ for every $(s_1, s_2)$ and for every $1 \leq i \leq n$. Then for $j = 1, 2$, $|\partial^j \theta_1| \leq K\kappa^{-(1+j)}$, and for $i \leq n$,*

$$|\partial^j (\theta_i - \theta_{i-1})| < (\frac{Kb}{\kappa^{(2+j)}})^{i-1}, \tag{4}$$

$$\|\partial^j M^{(i)} e_n\| < (\frac{Kb}{\kappa^{(2+j)}})^i. \tag{5}$$

Our next lemma is a perturbation result. Let $M_i, M'_i$ be two sequences of matrices, let $w$ be a vector, and let $\theta_i$ and $\theta'_i$ denote the angles $M^{(i)}w$ and $M'^{(i)}w$ make with the positive $x$-axis respectively.

**Lemma 2.2** ([BC2], Lemma 5.5) *Let $\kappa, \lambda$ be such that $\frac{Kb}{\kappa^2} < \lambda < K_0^{-12}\kappa^8$. If for $1 \le i \le n$, $\|M_i - M'_i\| \le \lambda^i$ and $\|M^{(i)}w\| \ge \kappa^i$, then*
*(a) $\|M'^{(n)}w\| \ge \frac{1}{2}\kappa^n$;*
*(b) $|\theta_n - \theta'_n| < \lambda^{\frac{n}{4}}$.*

Proofs of Lemmas 2.1, 2.2 and Corollary 2.2 are given in Appendix B.1.

**Hypothesis for Sects. 2.2 and 2.3:** $T : A \to A$ is an embedding of the form

$$T(x, y) = (t_1(x, y), bt_2(x, y))$$

where the $C^2$-norms of $t_1$ and $t_2$ are $\le K_0$, and $K_0 > 1$ and $b << 1$ are fixed numbers.

## 2.2 Stable curves

**Lemma 2.3** *Let $\kappa, \lambda$ be as in Lemma 2.2 and $z_0 \in A$ be such that for $i = 1, \cdots, n$, $\|DT^i(z_0)\| \ge \kappa^i$. Then there is a $C^1$ curve $\gamma_n$ passing through $z_0$ such that*
*(a) for all $z \in \gamma_n$, $d(T^i z_0, T^i z) \le (\frac{Kb}{\kappa^2})^i$ for all $i \le n$;*
*(b) $\gamma_n$ can be extended to a curve of length $\sim \lambda$ or until it meets $\partial A$.*

A proof of this lemma is given in Appendix B.2.

We call $\gamma_n$ a **stable curve of order** $n$. It will follow from this lemma that if $\|DT^i(z_0)\| \ge \kappa^i$ for all $i > 0$, then there is a **stable curve** $\gamma_\infty$ passing through $z_0$ obtained as a limit of the $\gamma_n$'s.

## 2.3 Curvature estimates

Let $\gamma_0 : [0, 1] \to A$ be a $C^2$ curve, and let $\gamma_i(s) = T^i(\gamma_0(s))$. We denote the curvature of $\gamma_i$ at $\gamma_i(s)$ by $k_i(s)$.

**Lemma 2.4** *Let $\kappa > b^{\frac{1}{3}}$. We assume that for every $s$, $k_0(s) \le 1$ and*

$$\|DT^j(\gamma_{n-j}(s))\gamma'_{n-j}(s)\| \ge \kappa^j \|\gamma'_{n-j}(s)\|$$

*for every $j < n$. Then*

$$k_n(s) \le \frac{Kb}{\kappa^3}.$$

A proof is given in Appendix B.3.

## 2.4 One-dimensional dynamics

We begin with some properties of maps satisfying the Misiurewicz condition. Let $f$ be as in Sect. 1.1, and let $C_\delta := \{x \in S^1 : d(x, C) < \delta\}$.

**Lemma 2.5** *There exist $\hat{c}_0, \hat{c}_1 > 0$ such that the following hold for all sufficiently small $\delta > 0$: Let $x \in S^1$ be such that $x, fx, \cdots, f^{n-1}x \notin C_\delta$, any $n$. Then*
*(i) $|(f^n)'x| \geq \hat{c}_0 \delta e^{\hat{c}_1 n}$;*
*(ii) if, in addition, $f^n x \in C_\delta$, then $|(f^n)'x| \geq \hat{c}_0 e^{\hat{c}_1 n}$.*

A proof is given in Appendix B.4.

**Corollary 2.3** *Let $c_0 < \hat{c}_0$ and $c_1 < \hat{c}_1$. Then for all sufficiently small $\delta$, there exists $\varepsilon = \varepsilon(\delta)$ such that for all $g$ with $\|g - f\|_{C^2} < \varepsilon$, (i) and (ii) above hold for $g$ with $c_0$ and $c_1$ in the places of $\hat{c}_0$ and $\hat{c}_1$.*

**Proof:** Let $N$ be such that $\delta e^{\hat{c}_1 N} > e^{c_1 N}$, and choose $\varepsilon$ small enough so that for all $i \leq N$, if $x, gx, \cdots, g^{i-1}x \notin C_\delta(g)$, then $(g^i)'x \approx (f^i)'x$. □

The results in the rest of this subsection are not needed in this article. We include them only as motivation for the corresponding results in 2-dimensions.

Temporarily write $C = C(g)$. To control $(g^n)'x$ when $g^i x \in C_\delta$ for some $i < n$, we need to impose further conditions on $g$. Following [BC1] and [BC2], we assume there exist $\lambda > 1$ and $0 < \alpha << 1$ such that for all $\hat{x} \in C$ and $n \geq 0$:

(a) $d(g^n \hat{x}, C) \geq c_0 e^{-\alpha n}$ and
(b) $|\ (g^n)'(g\hat{x})\ | \geq c_0 \lambda^n$.

We define for each $x \in C_\delta$ a **bound period** $p(x)$ as follows. Fix $\beta > \alpha$. Let $\hat{x} \in C$ be such that $|x - \hat{x}| < \delta$. Then $p(x)$ is the smallest $p$ such that

$$|g^p x - g^p \hat{x}| > c_0 e^{-\beta p}.$$

**Lemma 2.6 (Derivative recovery)** *There exists $K$ such that for $g$ satisfying the conditions above, if $|x - \hat{x}| = e^{-\mu} < \delta$ for some $\hat{x} \in C$, then*
*(i) $K^{-1}\mu \leq p(x) \leq K\mu$ ;*
*(ii) $K^{-1}(x - \hat{x})^2 |(g^{i-1})'(g\hat{x})| < |g^i x - g^i \hat{x}| < K(x - \hat{x})^2 |(g^{i-1})'(g\hat{x})|$;*
*(iii) $|(g^p)'x| \geq K^{-1} \lambda^{\frac{p}{2}}$ where $p = p(x)$.*

**Proof:** For this result there is no substantive difference between the situation here and that of the quadratic family $x \mapsto 1 - ax^2$. See [BC1] and [BC2], Section 2. □

**Standing hypotheses for the rest of the paper:** $\{T_{a,b}\}$ is as in Sect. 1.1. In particular, it has the form

$$T_{a,b}(x, y) \ = \ (F_a(x, y) + b u_{a,b}(x, y),\ b v_{a,b}(x, y)).$$

Where no ambiguity arises, we will write $T = T_{a,b}$. The phrase "for $(a, b)$ sufficiently near $(a^*, 0)$" will appear (finitely) many times in the next few sections. Each time it appears, the rectangle in parameter space for which our results apply may have to be reduced. From here on $K$ is the generic system constant as declared in Section 1.

## 2.5 Dynamics outside of $\mathcal{C}^{(0)}$

The first system constant to be chosen is $\delta$. A number of upper bounds for $\delta$ will be specified as we go along. For now we think of it as a very small positive number with $d(f^n\hat{x}, C) >> \delta$ for all $\hat{x} \in C$ and $n > 0$. We assume also that $a$ is sufficiently near $a^*$ that the Hausdorff distances between the critical sets of $f_{a^*}$ and $f_a$ are $<< \delta$.

Recall that we will be working in $R_0 = \{(x, y) \in A : |y| \leq Kb\}$. Our zeroth critical region $\mathcal{C}^{(0)}$ is defined to be

$$\mathcal{C}^{(0)} = \{(x, y) \in R_0 : |x - \hat{x}| < \delta \quad \text{for some } \hat{x} \in C\}.$$

Let $s(u)$ denote the slope of a vector $u$. Assuming that $b^{\frac{1}{5}} << \delta$, an easy calculation shows that for $z \notin \mathcal{C}^{(0)}$, if $|s(u)| < \delta^2$, then $|s(DT(z)u)| = \mathcal{O}(\frac{b}{\delta}) << \delta^2$. Also, if $\kappa_0 := \min \|DT(z)u\|$ where the minimum is taken over all $z \notin \mathcal{C}^{(0)}$ and unit vectors $u$ with $|s(u)| < \delta^2$, then $\kappa_0 > K^{-1}\delta$. Let $K(\delta) := \frac{K}{\kappa_0^3}$, so that $K(\delta)b$ is the upper bound for $k_n$ in Lemma 2.4. We call a vector $u$ a $b$**-horizontal vector** if $|s(u)| < K(\delta)b$. A curve $\gamma$ is called a $C^2(b)$**-curve** if its tangent vectors are $b$-horizontal and its curvature is $\leq K(\delta)b$ at every point.

**Lemma 2.7** *(a) For $z \notin \mathcal{C}^{(0)}$, if $u$ is $b$-horizontal, then so is $DT(z)u$.*
*(b) If $\gamma$ is a $C^2(b)$-curve outside of $\mathcal{C}^{(0)}$, then $T(\gamma)$ is again a $C^2(b)$-curve.*

**Proof:** (a) has already been explained; (b) is an immediate consequence of (a) and Lemma 2.4. $\square$

Our next lemma describes the dynamics of $b$-horizontal vectors outside of $\mathcal{C}^{(0)}$.

**Lemma 2.8** *There exist constants $c_0, c_1 > 0$ independent of $\delta$ such that the following holds for $T = T_{a,b}$ for all $(a, b)$ sufficiently near $(a^*, 0)$. Let $z \in R_0$ be such that $z, Tz, \cdots, T^{n-1}z \notin \mathcal{C}^{(0)}$, and let $u$ be a $b$-horizontal vector. Then*
*(i) $\|DT^n(z)u\| \geq c_0\delta e^{c_1 n}$;*
*(ii) if, in addition, $T^n z \in \mathcal{C}^{(0)}$, then $\|DT^n(z)u\| \geq c_0 e^{c_1 n}$.*

**Proof:** As with Corollary 2.3, this follows from Lemma 2.5 by perturbation. $\square$

## 2.6 Critical points inside $\mathcal{C}^{(0)}$

Wherever it makes sense, let $e_m$ denote the field of most contracted directions of $DT^m$ and let $q_m$ be the slope of $e_m$. When working with a curve $\gamma$ parameterized by arc length, we write $q_m(s) = q_m(\gamma(s))$. We begin with some easy observations about $e_1$.

**Lemma 2.9** *For all $(a,b)$ sufficiently near $(a^*,0)$, $e_1$ is defined everywhere on $R_0$, and there exists $K>0$ such that*

*(a) $|q_1| > K^{-1}\delta$ outside of $\mathcal{C}^{(0)}$, and $q_1$ has opposite signs on adjacent components of $R_0 \setminus \mathcal{C}^{(0)}$;*

*(b)*

$$|\frac{dq_1}{ds}| > K^{-1}$$

*on every $C^2(b)$-curve $\gamma$ in $\mathcal{C}^{(0)}$.*

**Proof:** The existence of $e_1$ follows from the fact that everywhere on $R_0$, $\|DT\| > K^{-1}$ (this uses the non-degeneracy condition in Step IV, Sect. 1.1) while $|\det(DT)| = \mathcal{O}(b)$. For $a=a^*$, $b=0$ and $\{y=0\}$, the assertion in (a) is obvious, and part (a) of Lemma 2.9 follows by a perturbative argument. The estimate for $|\frac{dq_1}{ds}|$ uses the non-degeneracy condition above and the fact that $f''_{a^*} \neq 0$ on $C$. See Appendix B.5 for details. □

**Definition 2.1** *Let $\gamma$ be a $C^2(b)$-curve in $\mathcal{C}^{(0)}$. We say that $z_0$ is a* **critical point of order** $m$ **on** $\gamma$ *if*

*(a) $\|DT^i(z_0)\| \geq 1$ for $i=1,2,\cdots,m$;*

*(b) at $z_0$, $e_m$ coincides with the tangent vector to $\gamma$.*

It follows from Lemma 2.9 that on every $C^2(b)$-curve that stretches across a component of $\mathcal{C}^{(0)}$, there is a unique critical point of order 1. The next two lemmas are used in the "updating" of existing critical points and the creation of new ones. Their proofs are given in Appendix B.5

**Lemma 2.10** ([BC2], p. 113) *Let $\gamma$ be a $C^2(b)$-curve in $\mathcal{C}^{(0)}$ where $\gamma(0)=z$ is a critical point of order $m$. We assume that*

*(a) $\|DT^i(z)\| \geq 1$ for $i=1,2,\cdots,3m$;*

*(b) $\gamma(s)$ is defined for $s \in [-(Kb)^{\frac{m}{2}}, (Kb)^{\frac{m}{2}}]$.*

*Then there exists a unique critical point $\hat{z}$ of order $3m$ on $\gamma$, and $|\hat{z}-z| < (Kb)^m$.*

**Lemma 2.11** ([BC2], Lemma 6.1) *For $\varepsilon > 0$, let $\gamma$ and $\hat{\gamma}$ be two disjoint $C^2(b)$-curves in $\mathcal{C}^{(0)}$ defined for $s \in [-4K_1\sqrt{\varepsilon}, 4K_1\sqrt{\varepsilon}]$ where $K_1$ is the constant $K$ in Lemma 2.9(b). We assume*

*(a) $\gamma(0)$ is a critical point of order $m$;*

*(b) the $x$-coordinates of $\gamma(0)$ and $\hat{\gamma}(0)$ coincide, and $|\,\gamma(0)-\hat{\gamma}(0)\,| < \varepsilon$.*

*Then there exists a critical point of order $\hat{m}$ at $\hat{\gamma}(\hat{s})$ with $|\hat{s}| < 4K_1\sqrt{\varepsilon}$ and $\hat{m} = min\{m, K\log\frac{1}{\varepsilon}\}$.*

## 2.7 Tracking $DT^n$: a splitting algorithm

The purpose of this section is to recall an algorithm introduced in [BC2] that gives, under suitable circumstances, a direct relation between $DT^n$ and 1-dimensional derivatives.

Let $z_0 \in R_0$, and let $w_0$ be a unit vector at $z_0$ that is $b$-horizontal. We write $z_n = T^n z_0$ and $w_n = DT^n(z_0)w_0$. In the case where $z_i \notin \mathcal{C}^{(0)}$ for all $i$, the resemblance to 1-d is made clear in Lemmas 2.5 and 2.8. Consider next an orbit $z_0, z_1, \cdots$ that visits $\mathcal{C}^{(0)}$ exactly once, say at time $t > 0$. Assume:

(a) there exists $\ell > 0$ such that $\|DT^i(z_t)\binom{0}{1}\| \geq 1$ for all $i < \ell$, so that in particular $e_\ell$, the most contracted direction of $DT^\ell$, is defined at $z_t$, and

(b) $\theta(w_t, e_\ell)$, the angle between $w_t$ and $e_\ell$, is $\geq b^{\frac{\ell}{2}}$.

Then $DT^i(z_0)$ can be analyzed as follows. (Note that our notation is different from that in [BC2].) We split $w_t$ into $w_t = \hat{w}_t + \hat{E}$ where $\hat{w}_t$ is parallel to the vector $\binom{0}{1}$ and $\hat{E}$ is parallel to $e_\ell$. For $i \leq t$ and $i \geq t + \ell$, let $w_i^* = w_i$. For $i$ with $t < i < t+\ell$, let $w_i^* = DT^{i-t}(z_t)\hat{w}_t$. We claim that all the $w_i^*$ are $b$-horizontal vectors, so that $\{\|w_{i+1}^*\|/\|w_i^*\|\}_{i=0,1,2,\cdots}$ resemble a sequence of 1-d derivatives. In particular, $\|w_{t+1}^*\|/\|w_t^*\| \sim \theta(w_t, e_\ell)$ simulates a drop in the derivative when an orbit comes near a critical point in 1-dimension.

To justify the statement about the slope of the $w_i^*$, we note that $DT(z_t)\binom{0}{1}$ is $b$-horizontal, so that in view of lemma 2.7 we need only to consider $w_{t+\ell}^*$. We have

$$\|DT^\ell(\hat{E})\| \leq b^\ell \frac{\|\hat{w}_t\|}{\theta(w_t, e_\ell)} \leq b^{\frac{\ell}{2}}\|\hat{w}_t\| \leq b^{\frac{\ell}{2}}\|DT^\ell(z_t)\hat{w}_t\|,$$

the first and third inequalities following from (a) and the second from (b). Since the slope of $DT^\ell(z_t)\hat{w}_t$ is smaller than $\frac{Kb}{2\delta}$, it follows that $w_{t+\ell}^* = DT^\ell(z_t)\hat{w}_t + DT^\ell(z_t)\hat{E}$ remains $b$-horizontal.

The discussion above motivates the following splitting algorithm introduced in [BC2]. Consider $\{z_i\}_{i=0}^\infty$, and let $t_1 < \cdots < t_j < \cdots$ be the times when $z_i \in \mathcal{C}^{(0)}$. We let $w_0$ be a $b$-horizontal unit vector, and assume as before that $e_{\ell_i}$ makes sense at $z_i$ for $i = t_j$. Define $w_i^*$ as follows:

1. For $0 \leq i \leq t_1$, let $w_i^* = DT^i(z_0)w_0$.
2. At $i = t_j$, we split $w_i^*$ into

$$w_i^* = \hat{w}_i + \hat{E}_i$$

where $\hat{w}_i$ is parallel to $\binom{0}{1}$ and $\hat{E}_i$ is parallel to $e_{\ell_i}$.

3. For $i > t_1$, let

$$w_i^* = DT(z_{i-1})\hat{w}_{i-1} \;+ \sum_{j:\; t_j+\ell_{t_j}=i} DT^{\ell_{t_j}}(z_{t_j})\hat{E}_{t_j} \tag{6}$$

and let $\hat{w}_i = w_i^*$ if $i \neq t_j$ for any $j$.

This algorithm does not give anything meaningful in general. It does, however, in the scenario of the next lemma.

**Lemma 2.12** *Let $z_i, w_i$ and $w_i^*$ be as above. Assume*

*(a) for each $i = t_j$, $\theta(w_i^*, e_{\ell_i}) \geq b^{\frac{\ell_i}{2}}$;*

*(b) the time intervals $I_j := [t_j, t_j + \ell_{t_j}]$ are strictly nested, i.e. for $j \neq j'$, either $I_j \cap I_{j'} = \emptyset$, $I_j \subset I_{j'}$, or $I_{j'} \subset I_j$, and $t_j + \ell_{t_j} \neq t_{j'} + \ell_{t_{j'}}$.*

*Then $w_i = w_i^*$ for $i \notin \cup_j I_j$, and the $w_i^*$'s are all $b$-horizontal vectors. The sequence $\{|w_i^*\|\}$ has the property that $\|w_{i+1}^*\|/\|w_i^*\| \sim \theta(w_i^*, e_{\ell_i})$ for $i = t_j$, and $\|w_{i+1}^*\| \approx \|DT(z_i)w_i^*\|$ for $i \neq t_j$.*

**Proof**: The nested condition in (b) allows us to consider the $I_j$'s one at a time beginning with the innermost time intervals. This reduces to the case of a single visit to $\mathcal{C}^{(0)}$ treated earlier on. □

# 3 The Critical Set

Many authors, including [BC1], [CE], [J], [M1], and [NS], have studied 1-dimensional maps by controlling their critical orbits. These ideas were mimicked in [BC2] where the authors developed techniques for identifying, for certain Hénon maps, a set they called the "critical set". This is done via an inductive procedure involving parameter selection. The first step in our analysis of the family $\{T_{a,b}\}$ is to carry out a similar parameter selection, and the aim of this section is to formulate suitable inductive hypotheses.

## 3.1 What is the critical set?

In 1-dimension, the critical set is where all previous expansion is destroyed. Tangencies of stable and unstable manifolds play a similar role in higher dimensions. Here is how we propose to capture the set $\mathcal{C}$ that we will prove in Section 7 to be the *origin* of all *nonhyperbolic behavior*.

Let $\mathcal{F}_0$ be the foliation on $R_0$ with leaves $\{y = \text{constant}\}$, and let $\mathcal{F}_k$ be its image under $T^k$. In Sect. 2.5 we defined the 0th critical region $\mathcal{C}^{(0)}$. Suppose that $T^i\mathcal{C}^{(0)} \cap \mathcal{C}^{(0)} = \emptyset$ for all $i \leq n_0$. Then for $i \leq n_0$, $\mathcal{F}_i$ restricted to $\mathcal{C}^{(0)} \cap R_i$ consists of finitely many bands of roughly horizontal leaves whose tangent vectors have been expanded the previous $i$ iterates (Lemma 2.8). From Corollaries 2.1, 2.2 and Lemma 2.9, we see also that in $\mathcal{C}^{(0)}$, $DT^i$ has a well-defined field of most contracted directions, namely $e_i$, whose integral curves are roughly parabolas. It is natural to take the set of tangencies in $\mathcal{C}^{(0)}$ between the leaves of $\mathcal{F}_i$ and the integral curves of $e_i$ to be our $i$th approximation of $\mathcal{C}$. Since these approximations stabilize quickly with $i$, they would

converge to $\mathcal{C}$ if this picture could be maintained indefinitely, i.e. if the "turns" of $\mathcal{F}_i$ could be prevented from entering $\mathcal{C}^{(0)}$ for all $i$.

This, however, is impossible. The "turns" in $\mathcal{F}_1$ generated by what corresponds to a single critical point of the 1-dimensional map form a 1-parameter family of parabolas whose vertices lie on a roughly horizontal curve. If this curve stays outside of $\mathcal{C}^{(0)}$, it expands exponentially and therefore must intersect $\mathcal{C}^{(0)}$ after a finite number of iterates. What comes to our rescue is the observation that the horizontal strips in $\mathcal{C}^{(0)} \cap R_i$ also become exponentially thin with $i$, so that in all likelihood a roughly vertical curve will intersect the attractor $\Omega$ in a very sparse Cantor set. Since it is the "turns" *inside* $\Omega$ that count, it suffices to consider a Cantor set of "turns", not the full 1-parameter family.

These observations suggest that we modify our strategy as follows. Since we do not know *a priori* the precise location of $\Omega$, it is natural to consider a sequence of curves that limit on $\Omega$, i.e. $\partial R_i$, $i = 1, 2, \cdots$. We replace $\mathcal{F}_0$ by $\partial R_0$, defining the $i$th approximation of the critical set for $i \le n_0$ to be the set of tangencies between $\partial R_i$ and the integral curves of $e_i$.

Experience from 1-dimension tells us that in order to retain a positive measure set of parameters, we must allow our "turns" to approach the critical set slowly. To maintain a picture similar to that for $i \le n_0$, we shrink the critical regions *sideways* at a rate faster than this rate of approach. As $i \to \infty$, the approximate critical sets converge to $\mathcal{C}$.

In order for the contractive fields above to be defined, it is necessary that the derivative along orbits starting from $\mathcal{C}$ experience some exponential growth. This growth, which is also useful for controlling the movements of the "turns", is brought about in two ways: (i) by arranging for $n_0$ in the first paragraph to be very large, growth is guaranteed for a long initial period; (ii) when an orbit of $\mathcal{C}$ gets near a point $z \in \mathcal{C}$, it copies the initial segment of the orbit of $z$, thereby *replicating* the growth properties created in (i).

A version of these ideas will be made precise in the inductive assumptions.

## 3.2 Getting started

The required initial growth in (i) above comes from the Misiurewicz property of $f$, the 1-dimensional map of which $T$ is a perturbation. By choosing $(a, b)$ sufficiently near $(a^*, 0)$ and $\delta$ sufficiently small, $n_0$ can be arranged to be arbitrarily large.

Let $\Gamma_0$ be the set of all critical points of order $n_0$ on $\partial R_0$. From Corollaries 2.1, 2.2 and Lemma 2.9, we know that each connected segment of $\partial R_0 \cap \mathcal{C}^{(0)}$ contains exactly one point of $\Gamma_0$. These are our **critical points of generation** 0.

In order to state properly our induction hypotheses, we introduce our main system constants. They are $\theta, \alpha, \beta, \rho$, $c$, and $n_0$ and $\delta$ (which we have met):

- There are two time scales, $N$ and a much slower one $\theta N$, where $\theta$ is chosen

so that $b^\theta = \mathcal{O}(1)$ and $< K^{-1}$ for some $K$ to be specified.

- $e^{-\alpha n}$ and $e^{-\beta n}$, with $\alpha << \beta << 1$, represent two small length scales.
- $c > 0$ is our target Lyapunov exponent; it is $< c_1$ where $c_1$ is as in Lemma 2.8.
- Finally, $0 < \rho < K^{-1}$ is an arbitrary number of order 1. It determines the rate at which our critical regions decrease in size (see Sect. 3.1).

The order in which these constants are chosen is as follows: $c, \rho, \alpha$ and $\beta$ are first fixed; $\delta$ is then taken as small as need to be. The last constants to be determined are $n_0$ and $\theta$; observe that $n_0 \to \infty$ and $\theta \to 0$ corresponds essentially to $(a, b) \to (a^*, 0)$.

Parameters are deleted at each stage of our induction. **Sections 3 –5 are concerned with the dynamics of the maps corresponding to the parameters retained. Issues pertaining to the measure of the set of retained parameters** (including whether or not it is nonempty) **are postponed to Section 6.**

## 3.3 Inductive assumptions

Let $N \geq n_0$ be a large number, and let $\Delta_N$ be the set of parameters retained after $N$ iterates. We now formulate a set of inductive assumptions that describes the desired dynamical picture for $T = T_{a,b}$, $(a, b) \in \Delta_N$. While we will continue to provide motivations and explanations, (IA1)–(IA6) below are to be viewed as formal inductive hypotheses. As before, let $z_i = T^i z_0$.

### 3.3.1 Critical points and critical regions

**(IA1) (Structure of critical regions)** *For all $k \leq \theta N$, the critical regions $\mathcal{C}^{(k)}$ are defined and have the geometric properties stated in* (1)(i), (ii) *and* (iii) *of Theorem 1. Moreover, on each horizontal boundary of each component of $\mathcal{C}^{(k)}$, there is a critical point of order $N$ located within $\mathcal{O}(b^{\frac{k}{3}})$ of the midpoint of the segment.*

Critical points on $\partial\mathcal{C}^{(k)}$ are called **critical points of generation** $k$. The set of critical points of generation $\leq k$ is denoted by $\Gamma_k$.

### 3.3.2 Distance to critical set and loss of hyperbolicity

If the critical set is where would-be stable and unstable directions are interchanged, then distance to the critical set might provide a measure of loss of hyperbolicity. This is indeed the case under suitable circumstances and for a suitable notion of “distance”.

If $Q$ is a component of $\mathcal{C}^{(k)}$, we let $L_Q$ denote the vertical line midway between the two vertical boundaries of $Q$.

**Definition 3.1** *We say $z \in \mathcal{C}^{(0)}$ is* **horizontally related** *or simply* **h-related** *to $\Gamma_{\theta N}$ if there exists a component $Q$ of $\mathcal{C}^{(k)}$, $k \leq \theta N$, such that $z \in Q$ and $dist(z, L_Q) \geq b^{\frac{k}{20}}$. When this holds, we say $z$ is h-related to $z_0$ for all $z_0 \in \Gamma_{\theta N} \cap Q$.* [6]

This is an attempt to describe the location of a point relative to $\Gamma_{\theta N}$, which, as $N \to \infty$, converges to a fractal set. From Lemma 4.1, we see that $\Gamma_{\theta N} \cap Q$ is contained in a region of width $\mathcal{O}(b^{\frac{k}{4}})$ in the middle of $Q$, so that $z$ and $\Gamma_{\theta N} \cap Q$ have a very obviously horizontal relationship. We caution, however, that there may be points in $\Gamma_{\theta N}$ that are directly above or below $z$, and quite possibly both to its left and to its right. Observe also that if $Q'$ is a component of $\mathcal{C}^{(k')}$ such that $z \in Q' \subset Q$, then $dist(z, L_{Q'}) \geq b^{\frac{k'}{20}}$.

**Definition 3.2** *For $z \in R_0$, we define its* **distance to the critical set***, denoted $d_{\mathcal{C}}(z)$, as follows: for $z \notin \mathcal{C}^{(0)}$, let $d_{\mathcal{C}}(z) = \delta$; for $z \in \mathcal{C}^{(0)}$, we let $d_{\mathcal{C}}(z) = dist(z, L_Q)$ where $Q$ is the component of $\mathcal{C}^{(k)}$ containing $z$ and $k$ is the largest number $\leq \theta N$ with $z \in \mathcal{C}^{(k)}$. We let $\phi(z)$ be one of the two points in $\partial Q \cap \Gamma_{\theta N}$ if $z$ is h-related to $\Gamma_{\theta N}$.*

For $z \in \mathcal{C}^{([\theta N])}$, the definitions of $d_{\mathcal{C}}(z)$ and $\phi(z)$ are temporary and will be modified as the induction progresses.

To secure growth properties for the orbits of $\Gamma_{\theta N}$, we forbid them to approach the critical set too closely too soon. (IA2) is a result of parameter selection.

**(IA2) (Rate of approach to critical set)** *For all $z_0 \in \Gamma_{\theta N}$ and all $i \leq N$, $d_{\mathcal{C}}(z_i) \geq min(\delta, e^{-\alpha i})$.*

(IA2) implies that for all $z_0 \in \Gamma_{\theta N}$ and $i \leq N$, $z_i$ is h-related to $\Gamma_{\theta N}$ whenever it is in $\mathcal{C}^{(0)}$. Intuitively, this is because $z_i$ is in a very "deep" layer relative to its distance to $\Gamma_{\theta N}$. Formally, let $z \in Q \subset \mathcal{C}^{(k)}$ where $Q$ and $k$ are as in Definition 3.2. Then $k << i$ since $\rho^k \geq e^{-\alpha i}$. Now $z_i \in R_i$. If $k < [\theta N]$, then $z_i \in Q \cap R_{k+1}$, proving $d_{\mathcal{C}}(z_i) \geq \rho^{k+1} >> b^{\frac{k}{20}}$. If $k = [\theta N]$, then $d_{\mathcal{C}}(z_i) \geq e^{-\alpha i} \geq e^{-\alpha N} >> b^{\frac{1}{20}\theta N}$ provided that $b^\theta$ is chosen to be $< e^{-20\alpha}$.

**Definition 3.3** *(a) For arbitrary $z \in \mathcal{C}^{(0)}$, we define its* **fold period** *$\ell(z)$ to be the nonnegative integer $\ell \geq 1$ such that $b^{\frac{\ell}{2}}$ is closest to $d_{\mathcal{C}}(z)$.*

*(b) Given $z_0 \in R_0$ and unit vector $w_0$, we let $w_i^*$, $i = 0, 1, 2, \cdots$, be given by the splitting algorithm in Sect. 2.7 with $\ell_i = \ell(z_i)$ assuming $e_{\ell(z_i)}$ is defined at $z_i$.*

Recall that for $z_0 \in \Gamma_{\theta N}$, $\|DT^i(z_0)\| \geq 1$ for all $i \leq N$. For $\ell \leq N$, Lemma 2.2 gives an estimate on the size of the neighborhood of $\Gamma_{\theta N}$ on which $e_\ell$ is well defined. In particular, if $z$ is h-related to $\Gamma_{\theta N}$, then $e_{\ell(z)}$ is defined at $z$.

[6] When studying the dynamics of $T$ on $\partial R_k$, it will be convenient to include the following in the definition of h-relatedness: Let $\gamma$ be a horizontal boundary of a component of $\mathcal{C}^{(k)}$, $k \leq \theta N$, and let $\hat{z} \in \gamma \cap \Gamma_{\theta N}$. Then $z \in \gamma$ is also said to be h-related to $\hat{z}$.

We fix $\varepsilon_0 > 0$ such that $\varepsilon_0 << |\frac{\partial q_1}{\partial x}|$ in $\mathcal{C}^{(0)}$ where $q_1$ is the slope of $e_1$. For $z \in \partial R_k$, let $\tau(z)$ denote a tangent vector to $\partial R_k$ at $z$. In the angle estimates below, $\tau$ and $e_\ell$ are assumed to point in roughly the same direction as $w$.

**Definition 3.4** *Let $z \in \mathcal{C}^{(0)}$ be h-related to $\Gamma_{\theta N}$, and let $w$ be a vector at $z$. We say $w$* **splits correctly** *if $|\frac{w}{\|w\|} - \tau(\phi(z))| < \varepsilon_0 d_{\mathcal{C}}(z)$.*

**(IA3) (Correct splitting at returns)** *For $z_0 \in \Gamma_{\theta N}, w_0 = \binom{0}{1}$ and $i \leq N$, $w_i^*$ splits correctly whenever $z_i \in \mathcal{C}^{(0)}$.*

The sense in which this splitting is "correct" is as follows. We wish to use Lemma 2.12 to understand the evolution of $w_i$. First, (IA3) and Lemma 2.9 together imply condition (a) of this lemma. This is because $|e_{\ell_i}(z_i) - \frac{w_i^*}{\|w_i^*\|}| \geq |e_{\ell_i}(z_i) - e_{\ell_i}(\phi(z_i))| - |e_{\ell_i}(\phi(z_i)) - \tau(\phi(z_i))| - |\tau(\phi(z_i)) - \frac{w_i^*}{\|w_i^*\|}| \geq |\frac{\partial q_{\ell_i}}{\partial x}| d_{\mathcal{C}}(z_i) - \mathcal{O}(b^{\ell_i}) - \varepsilon_0 d_{\mathcal{C}}(z_i) \geq \frac{1}{2}|\frac{\partial q_1}{\partial x}| d_{\mathcal{C}}(z_i) \sim b^{\frac{\ell_i}{2}}$. (For a comparison of $|\frac{\partial q_{\ell_i}}{\partial x}|$ and $|\frac{\partial q_1}{\partial x}|$, see Corollary 2.2.) Condition (b) of Lemma 2.12 is discussed in Sect. 4.1.

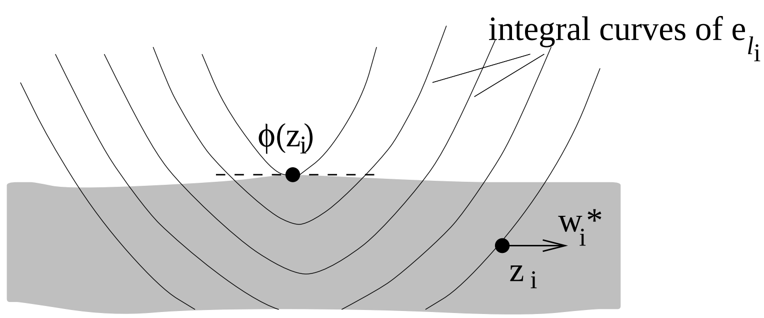



Figure 2 Correct splitting of $w_i^*$

### 3.3.3 Derivative along critical orbits

We saw in the last paragraph that for $z_0 \in \Gamma_{\theta N}$, as $z_i$ enters $\mathcal{C}^{(0)}$, $w_i^*$ suffers a loss of hyperbolicity proportional to $d_{\mathcal{C}}(z_i)$. Combining this with (IA5)(c) below applied to an earlier step, we see that this loss will be partially – but not fully – compensated for at the end of a certain period. To prevent a downward spiral in Lyapunov exponent, further parameter exclusion is needed.

**(IA4) (Derivative growth)** *For all $z_0 \in \Gamma_{\theta N}$ and $0 \leq i \leq \frac{2}{3}N$, $\|w_i^*(z_0)\| > c_0 e^{ci}$.*

In future steps of the induction, orbits of length $\frac{2}{3}N$ starting from $\Gamma_{\theta N}$ will be replicated; in other words, they will serve as guides for other points that enter $\mathcal{C}^{(0)}$.

**Definition 3.5** *For arbitrary* $\xi_0$ *and* $\xi_0' \in \mathcal{C}^{(0)}$*, we define their* **bound period** *to be the largest integer* $p$ *such that for all* $0 < j \le p$*,*

$$|\xi_j - z_j| \ \le \ e^{-\beta j}.$$

Observe that if $\xi_0' = z_0 \in \Gamma_{\theta N}$, then for $j \le p$, $|\xi_j - z_j| \ << d_{\mathcal{C}}(z_j)$. We may assume $\delta$ is so small and $n_0$ so large that $d_{\mathcal{C}}(\xi_j) > \frac{\delta}{2}$ when $z_j$ is outside of $\mathcal{C}^{(0)}$. Our last two inductive assumptions deal with the properties $z_0$ passes along to $\xi_0$.

**(IA5) (Similarities with 1-dimensional maps)** *Let* $z_0 \in \Gamma_{\theta N} \cap \partial\mathcal{C}^{(k)}$*, and let* $\gamma : [0, \varepsilon] \to \mathcal{C}^{(0)}$ *be a* $C^2(b)$*-curve with* $\gamma(0) = z_0$ *and* $\gamma'(0)$ *tangent to* $\partial\mathcal{C}^{(k)}$*. We regard all* $\xi_0 \in \gamma$ *as bound to* $z_0$*, and let* $p(\xi_0)$ *denote their bound periods. Then:*

*(a) There exists* $K$ *such that for* $\xi_0 \in \gamma$ *with* $|\xi_0 - z_0| = e^{-h}$*,*

$$\frac{1}{K}h \le p(\xi_0) \le Kh \quad \textit{provided} \quad Kh < \frac{2}{3}N;$$

*moreover,* $p(\xi_0)$ *increases monotonically with the distance between* $\xi_0$ *and* $z_0$*;*

*(b) for* $\ell \le j \le min(p, \frac{2}{3}N)$, $|\xi_j - z_j| \approx |\xi_0 - z_0|^2 \|w_j(z_0)\|$ *where "*$\approx$*" means up to a factor of* $(1 \pm \varepsilon_1)$ *for some* $\varepsilon_1 > 0$*;*

*(c)* $\|w_p(\xi_0)\| \cdot |\xi_0 - z_0| \ge e^{\frac{cp}{3}}$ *provided* $p < \frac{2}{3}N$*.*

(IA5) describes the quadratic nature of the "turn" as $\gamma$ is mapped forward. For comparison with 1-dimensional behavior, see Lemma 2.6.

The following distortion estimates are used in the proof of (IA5). Let $w_0(\xi_0) = w_0(z_0) = \binom{0}{1}$, and let $\hat{w}_i^*(\xi_0)$ be given by Definition 3.3(b) except that $e_{\ell(z_i)}$ (and not $e_{\ell(\xi_i)}$) is used for splitting at time $i$. (IA6) compares $w_i^*(z_0)$ and $\hat{w}_i^*(\xi_0)$. Let $M_i(\cdot)$ and $\theta_i(\cdot)$ denote the magnitude and argument of the vectors in question. Define

$$\Delta_i(\xi_0, z_0) = \sum_{s=0}^{i} (Kb)^{\frac{s}{4}} \mid \xi_{i-s} - z_{i-s} \mid . \tag{7}$$

**(IA6) (Distortion bounds)** *Given* $z_0 \in \Gamma_{\theta N}$ *and any* $\xi_0 \in \mathcal{C}^{(0)}$*, we regard* $\xi_0$ *as bound to* $z_0$ *and let* $p$ *be the bound period. Then for* $i \le min\{p, N\}$*,*

$$\frac{M_i(z_0)}{M_i(\xi_0)}, \ \frac{M_i(\xi_0)}{M_i(z_0)} \ \le \ \exp\{K \sum_{j=1}^{i-1} \frac{\Delta_j}{d_{\mathcal{C}}(z_j)}\} \tag{8}$$

*and*

$$\mid \theta_i(\xi_0) - \theta_i(z_0) \mid \le (Kb)^{\frac{1}{2}} \Delta_{i-1}. \tag{9}$$

*The estimates above also hold with* $w_i^*(z_0)$ *replaced by* $\hat{w}_i^*(\xi_0')$ *where* $\xi_0'$ *is another point in* $\mathcal{C}^{(0)}$ *also thought of as bound to* $z_0$*, and* $p$ *is the minimum of the two bound periods.*

Let us return for a moment to Definition 6. From the geometry of $\mathcal{C}^{(k)}$ (see (IA1) and Lemma 4.1) it is an exercise in calculus to show that if $\xi_0$ is h-related to $z_0 \in \Gamma_{\theta N}$, then it lies on a $C^2(b)$-curve through $z_0$ tangent to $\tau(z_0)$. In particular, (IA5) applies.

Our rules of parameter exclusion, namely (IA2) and (IA4), are similar to those used in [BC2], but they are applied to different orbits and with a different definition of "$d_{\mathcal{C}}(\cdot)$". The notions of bound and fold periods are borrowed from [BC2], as are (IA5) and (IA6). Our construction of $\mathcal{C}$, however, has a distinctly different flavor.

# 4 Replication of Orbit Segments

In Sect. 3.1 we outlined a scheme for obtaining derivative growth along critical orbits, namely to choose a start-up geometry that guarantees some initial growth, and then to try to replicate this behavior. Section 4 contains a detailed analysis of the replication process. The main results are stated in Sect. 4.3, after some technical preparations in Sects. 4.1 and 4.2, including amending slightly the definitions of bound and fold periods. (IA1)–(IA6) are assumed up to time $N$.

## 4.1 Nested properties of bound and fold periods

Consider $z_0 \in \Gamma_{\theta N}$. When $z_i$ enters $\mathcal{C}^{(0)}$, it is natural to assign to it a **bound period** $p(z_i)$ defined using $\phi(z_i)$. An unsatisfactory aspect of this definition is that two bound periods so defined may overlap without one being completely contained in the other. The purpose of this subsection is to adjust slightly the definition of $p(z_i)$ to create a simpler binding structure. A similar adjustment is made in [BC2].

First we fix some notation. Let $Q^{(j)}$ denote the components of $\mathcal{C}^{(j)}$, and let $\hat{Q}^{(j)}$ be the component of $R_j \cap \mathcal{C}^{(j-1)}$ containing $Q^{(j)}$. For $z \in \partial R_j$, let $\tau(z)$ be a unit vector at $z$ tangent to $\partial R_j$.

**Lemma 4.1** *For $z, z' \in \Gamma_{\theta N} \cap Q^{(k)}$, we have*

$$|z - z'| = \mathcal{O}(b^{\frac{k}{4}}) \quad \text{and} \quad \|\tau(z) \times \tau(z')\| = \mathcal{O}(b^{\frac{k}{4}}).$$

**Proof:** Let $z^{(k)}$ be a critical point in $\partial Q^{(k)}$. For $k \le i < [\theta N]$, let $z^{(i+1)}$ be a critical point of generation $i+1$ in $Q^{(i)}(z^{(i)})$, the component of $Q^{(i)}$ containing $z^{(i)}$. From (IA1) we know that the Hausdorff distance between the two horizontal boundaries of $Q^{(i)}(z^{(i)})$ is $\mathcal{O}(b^{\frac{i}{2}})$. Lemma 2.11 then tells us that $|z^{(i)} - z^{(i+1)}| = \mathcal{O}(b^{\frac{i}{4}})$. The angle estimate also follows from the proof of Lemma 2.11 □

**Lemma 4.2** *Let $\xi_0$ be h-related to $z_0 \in \Gamma_{\theta N}$. If during their bound period $z_i$ returns to $\mathcal{C}^{(k)}$, then $\xi_i \in \hat{Q}^{(k)}(z_i)$.*

**Proof:** Let $\gamma$ be a $C^2(b)$-curve joining $z_0$ and $\xi_0$. Then $T^i\gamma \subset R_i$. Since $e^{-\alpha i} \le d_{\mathcal{C}}(z_i) \le \rho^k$, we have $k < i$ and therefore $T^i\gamma \subset R_k$. By the monotonicity of bound periods, every point in $T^i\gamma$ is within a distance of $< e^{-\beta i}$ from $z_i$. This puts $\xi_i \in R_k \cap Q^{(k-1)}(z_i)$. □

**Lemma 4.3** *Let $z_0 \in \Gamma_{\theta N}$ be such that $z_i \in \mathcal{C}^{(0)}$ at times $t_1 < t_2 < \cdots < t_r$, and that for each $j < r$ the bound period $p_j$ initiated at time $t_j$ extends beyond time $t_{j+1}$. Then $p_j < (K\alpha)^{j-1}p_1$.*

**Proof:** Let $\tilde{z}_0 = \phi(z_{t_1})$. We claim that $|z_{t_2} - \phi(z_{t_2})| \approx |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$, which is $> e^{-\alpha(t_2-t_1)}$. If true, this will imply, by (IA5)(a), that $p_2 < K\alpha(t_2 - t_1) < K\alpha p_1$, and

the assertion in the lemma will follow inductively. Since $|z_{t_2} - \tilde{z}_{t_2-t_1}| < e^{-\beta(t_2-t_1)} <<$ $e^{-\alpha(t_2-t_1)}$, it suffices to show that $|\phi(\tilde{z}_{t_2-t_1}) - \phi(z_{t_2})| << |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$. Let $k$ be the largest number such that $\tilde{z}_{t_2-t_1} \in \mathcal{C}^{(k)}$. By Lemma 4.2, $z_{t_2} \in Q^{(k-1)}(\tilde{z}_{t_2-t_1})$, so $\phi(\tilde{z}_{t_2-t_1})$ and $\phi(z_{t_2})$ must both be in $Q^{(k-1)}(\tilde{z}_{t_2-t_1})$. By Lemma 4.1 they are $\leq b^{\frac{k-1}{4}}$ apart, and this is $<< |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$. $\square$

**Definition 4.1** *For $z_0 \in \Gamma_{\theta N}$ with $z_i \in \mathcal{C}^{(0)}$, the* **adjusted bound period** *$p^*(z_i)$ is defined to be the smallest number $p^*$ with the property that for all $j$ with $i \leq j < i+p^*$, if $z_j \in \mathcal{C}^{(0)}$, then $j + p(z_j) \leq i + p^*$.*

Adjusted bound periods, therefore, have a nested structure by definition.

**Corollary 4.1** *(a) $p^* \leq p + K\alpha p$.*
*(b) For $z_i \in \mathcal{C}^{(0)}$ with $\phi(z_i) = \hat{z}_0$, we have for all $j \leq p^*$,*

$$|z_{j+i} - \hat{z}_j| < e^{-\beta^* j}$$

*for some $\beta^*$ smaller than $\beta$ and $>> \alpha$.*

The proof is left as an exercise. We assume from here on that all bound periods for all critical orbits are adjusted, and write $p$ and $\beta$ instead of $p^*$ and $\beta^*$.

This amended definition gives critical orbits the following simple structure of **bound** and **free states**. We call $z_i$ a **return** if $z_i \in \mathcal{C}^{(0)}$. Then $z_i$ is free for $i \leq n_1$ where $n_1 > 0$ is the time of the first return, and it is in bound state for $n_1 < i \leq n_1+p_1$ where $p_1$ is the bound period initiated at time $n_1$. After time $n_1 + p_1$, $z_i$ remains free until its next return at time $n_2$, is bound for the next $p_2$ iterates, and so on. The times $n_j$ are called **free return** times. A **primary bound period** begins at each $n_j$. Inside the time interval $[n_j, n_j + p_j]$, there may be **secondary bound periods** which comprise disjoint time intervals, and so on.

Next we consider fold periods, which are denoted by $\ell$ and defined in Sect. 3.3.2. As with bound periods, if $z_i$ enters $\mathcal{C}^{(0)}$ at times $t_1$ and $t_2$ with $t_1 < t_2 \leq N$, and if the fold period begun at $t_1$ remains in effect at $t_2$, then using Lemma 4.2 we see that $\ell_{t_2} < \frac{\alpha}{\log \frac{1}{b}} \ell_{t_1}$, so that **adjusted fold periods** can be defined similarly to give a nested structure. This is condition (b) of Lemma 2.12 . A further simplifying arrangement, which we will also adopt, is that no fold periods expire at returns to $\mathcal{C}^{(0)}$ or at the step immediately after. The proof of the following lemma is straightforward and will be omitted.

**Lemma 4.4** (cf. [BC2], Lemma 6.5) *Let $z_0 \in \Gamma_{\theta N}$. Then for every $i < N$, there exist $i_1 \leq i \leq i_2$ with*

$$i_2 - i_1 < K\theta\alpha i$$

*such that $i_1$ and $i_2$ are out of all fold periods.*

## 4.2 Orbits controlled by $\Gamma_{\theta N}$

In this subsection we consider $(z_0, w_0)$ where $z_0$ is an arbitrary point in $R_0$ and $w_0$ is a unit vector. We write $z_i = T^i z_0$ and $w_i = DT^i(z_0)w_0$.

**Definition 4.2** *We say $(z_0, w_0)$ is* **controlled** *by $\Gamma_{\theta N}$ up to time $m$ (with $m$ possibly $> N$) if the following hold.*

- *Initial conditions: if $z_0 \notin \mathcal{C}^{(0)}$, then $w_0$ is a b-horizontal vector; if $z_0 \in \mathcal{C}^{(0)}$, then either $w_0 = \binom{0}{1}$, or $z_0$ is h-related to $\Gamma_{\theta N}$ and $w_0$ splits correctly.*
- *For $0 < i \le m$, if $z_i \in \mathcal{C}^{(0)}$, then $z_i$ is h-related to $\Gamma_{\theta N}$ and $w_i^*$ splits correctly.*

No h-relatedness property is required for $z_0 \in \mathcal{C}^{(0)}$ when $w_0 = \binom{0}{1}$ because for practical purposes, one may think of the sequence as starting with $(z_1, w_1)$.

Let $(z_0, w_0)$ be as above. Then the orbit of $z_0$ has a natural bound/free structure defined as follows: If $z_0 \in \Gamma_{\theta N}$, then it is natural to regard $z_0, z_1, \cdots, z_i$ as free until $z_i$ returns to $\mathcal{C}^{(0)}$. For $z_0 \in \mathcal{C}^{(0)} \setminus \Gamma_{\theta N}$, we may regard $z_0$ as bound to any $\hat{z} \in \Gamma_{\theta N}$ for a period $p$ provided that $(\max \|DT\|)^p\ |z_0 - \hat{z}| < e^{-\beta p}$. (This trivial bound period is used to ensure that Lemma 4.2 continues to work.) When $z_i$ is h-related to $\Gamma_{\theta N}$, we take the bound period to be that between $z_i$ and $\phi(z_i)$ (which is longer than the trivial one). Observe that Lemma 4.3 is equally valid for controlled orbits as for orbits starting from $\Gamma_{\theta N}$, so that a nested structure can also be assumed for the bound and fold periods of controlled orbits.

In the language of Definition 4.2, the situation can be summed up as follows. First, it follows from (IA2) and (IA3) that for all $\hat{z}_0 \in \Gamma_{\theta N}$, $(\hat{z}_0, \binom{0}{1})$ is controlled by $\Gamma_{\theta N}$ up to time $N$. Second, for $(z_0, w_0)$ controlled by $\Gamma_{\theta N}$, (IA5) and (IA6) apply to give information during its bound periods. In particular, the orbit of $(z_0, w_0)$ has similar bound/free structures and "derivative recovery" estimates as those of $(\hat{z}_0, \binom{0}{1})$, $\hat{z}_0 \in \Gamma_{\theta N}$, except that (IA2) and (IA4) need not hold.

In the remainder of this subsection we record some technical facts on $w_i$ and $w_i^*$. Their proofs are given in Appendix B.6. In Lemmas 4.6–4.8, it is assumed that $(z_0, w_0)$ is controlled by $\Gamma_{\theta N}$ up to time $m$, and all time indices are $\le m$.

**Lemma 4.5** *Suppose $(z_0, w_0)$ satisfies the initial conditions in Definition 4.2, and for $0 < i \le m$, $z_i$ is h-related to $\Gamma_{\theta N}$ at all returns. Then $(z_0, w_0)$ is controlled up to time $m$ if $w_i^*$ splits correctly at all free returns.*

**Lemma 4.6** *Under the additional assumption that $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $i \le m$, we have*

$$K^{-\varepsilon i}\|w_i^*\| \ \le\ \|w_i\| \ \le\ K^{\varepsilon i} e^{\alpha i}\|w_i^*\|, \qquad \varepsilon = K\alpha\theta.$$

**Lemma 4.7** *There exists $c' > 0$ such that for every $0 \le k < n$,*

$$\|w_n^*\| \ge K^{-1} d_{\mathcal{C}}(z_j) e^{c'(n-k)} \|w_k^*\|$$

*where $j$ is the first time $\ge k$ when a bound period extending beyond time $n$ is initiated. If no such $j$ exists, set $d_{\mathcal{C}}(z_j) = 1$.*

**Lemma 4.8** *Let $k < n$ and assume $z_n$ is free. Then*

$$\|w_n\| > K^{-K\theta(n-k)} e^{c'(n-k)} \|w_k\|.$$

## 4.3 Controlled orbits as "guides" for other orbits

(IA2)–(IA6) are about orbits starting from $\Gamma_{\theta N}$. In Sect. 4.2 we introduced a class of orbits that successfully use orbits from $\Gamma_{\theta N}$ as their "guides". We now let these orbits serve as guides for other orbits and study the properties they pass along. This is the essence of the replication process.

Throughout Sect. 4.3 we assume that

(1) $z_0 \in \mathcal{C}^{(0)}$, $w_0 = \binom{0}{1}$, and $(z_0, w_0)$ is controlled by $\Gamma_{\theta N}$ up to time $m$;
(2) $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $0 < i \le m$.

Our first order of business is to establish that for all $\xi_0$ bound to $z_0$, $\hat{w}_i^*(\xi_0)$ copies $w_i^*(z_0)$ faithfully. A detailed proof of the following lemma is given in Appendix B.7.

**Lemma 4.9** (cf. [BC2], Lemma 7.8) *Let $(z_0, w_0)$ be as above, and let $\xi_0 \in \mathcal{C}^{(0)}$ be an arbitrary point which we think of as bound to $z_0$. Let $M_\mu(\cdot)$ and $\theta_\mu(\cdot)$ have the same meaning in (IA6). Then the estimates for*

$$\frac{M_\mu(\xi_0)}{M_\mu(z_0)}, \quad \frac{M_\mu(z_0)}{M_\mu(\xi_0)} \quad \textit{and} \quad |\theta_\mu(\xi_0) - \theta_\mu(z_0)|$$

*as stated in (IA6) hold for all $\mu \le min(p, m)$. The corresponding distortion estimates for two points $\xi_0$ and $\xi_0'$ bound to $z_0$ apply as well.*

In the rest of this subsection we consider the situation where $z_0$ is a critical point on a $C^2(b)$-curve in the sense of Sect. 2.6 and study the quadratic behavior as this curve is iterated. More precisely, let $e_m$ be the contractive field of order $m$, which we know from Lemmas 4.6 and 4.7 is defined at $z_0$. We assume

(3) $z_0$ lies on a $C^2(b)$-curve $\gamma \subset \mathcal{C}^{(0)}$, and $e_m(z_0)$ is tangent to $\gamma$.

For $\xi_0 \in \gamma$, let $p = p(\xi_0)$ denote the bound period between $z_0$ and $\xi_0$. We assume that during its bound period, the orbit of $\xi_0$ inherits the secondary and higher order bound structures of the orbit of $z_0$.

**Lemma 4.10** *In the part of $\gamma$ where $p < m$, $p$ increases monotinically with distance from $z_0$.*

**Proof:** Proceeding inductively, we assume that on a connected subsegment $\gamma_k$ of $\gamma$ one of whose end points is $z_0$, the minimum bound period is $k$. It suffices to show that at time $k+1$, the part of $\gamma_k$ that remains bound to $z_0$ is connected. We may assume $T^k(\gamma_k)$ is not in a secondary fold period (otherwise all of $T^{k+1}(\gamma_k)$ will be in a bound period), and that $d_{\mathcal{C}}(\xi_0) > \frac{1}{2}\delta$ for all $\xi_0 \in T^k(\gamma_k)$.

Let $T^k(\gamma_k) = \gamma^{(1)} \cup \gamma^{(2)}$ where $\gamma^{(1)}$ consists of points for which the primary fold period remains in effect and $\gamma^{(2)}$ its complement. Then $\gamma^{(1)}$ is contained in a disk $B$ of radius $K^k b^{\frac{k}{2}}$ centered at $z_k$, and the bound period on no part of $B$ can expire at time $k+1$. If the bound period of any part of $\gamma^{(2)}$ is to expire at time $k+1$, then the far end of $\gamma^{(2)}$ must be $> K^{-1}e^{-\beta(k+1)}$ from $z_k$. Also, its tangent vectors are $b$-horizontal. One concludes that $T^k(\gamma) \setminus B$ is a $b$-horizontal connected segment which will remain horizontal in the next iterate, forcing the desired picture. $\square$

Let $s \to \xi_0(s)$ be the parametrization of $\gamma$ by arc length with $\xi_0(0) = z_0$. The following lemma, whose proof is given in Appendix B.8, contains a **distance formula** for $|\xi_\mu(s) - z_\mu|$. See Sect. 2.4 for comparison with 1-d.

**Lemma 4.11** *Let $\varepsilon_1 > 0$ be given. Then assuming $\delta$ is sufficiently small, we have, for all $\mu \in \mathbb{Z}^+$ and $s > 0$ satisfying $\mu \le m$, $(Kb)^{\frac{\mu}{2}} < s$ and $p(\xi_0(s)) \ge \mu$,*

$$(1-\varepsilon_1)\ \|w_\mu(0)\|\ K_1 s^2\ <\ |\xi_\mu(s) - z_\mu|\ <\ (1+\varepsilon_1)\ \|w_\mu(0)\|\ K_1 s^2 \tag{10}$$

*where $K_1 = \frac{1}{2}|\frac{dq_1}{dx}(z_0)|$.*

**Corollary 4.2** *Assume in addition to (1)–(3) above that $\|w_j^*(z_0)\| > e^{cj}$ for all $j \le m$. Let $\xi_0 \in \gamma$. Suppose that $|\xi_0 - z_0| = e^{-h}$ and $p(\xi_0) \le m$. Then*
*(a) $\frac{h}{3K_2} \le p \le \frac{3h}{c}$ where $K_2 = \log\|DT\|$;*
*(b) $\|w_p(\xi_0)\| \cdot |\xi_0 - z_0| \ge e^{\frac{cp}{3}}$.*

**Proof:** (a) The lower bound for $p$ follows from the fact that for all $j \le \frac{h}{3K_2}$, $|\xi_j - z_j| < \|DT\|^j|\xi_0 - z_0| < e^{-\frac{2h}{3}} << e^{-\beta\frac{h}{3K_2}}$. By Lemma 4.11, $p$ is the smallest $\mu$ such that $\|w_\mu(0)\| \cdot |z_0 - \xi_0|^2 > K_1^{-1}e^{-\beta\mu}$. This must happen for some $\mu \le \frac{3h}{c}$ because $\|w_{\frac{3h}{c}}(z_0)\| \cdot |z_0 - \xi_0|^2 > K^{-\varepsilon\frac{3h}{c}}\|w^*_{\frac{3h}{c}}(z_0)\| \cdot |z_0 - \xi_0|^2 > K^{-\varepsilon\frac{3h}{c}}e^{c\cdot\frac{3h}{c}}e^{-2h} > 1$.

(b) This follows from the fact that $\|w_p(\xi_0)\| \approx \|w_p(z_0)\|$ (Lemma 4.9) and $|z_0 - \xi_0| \cdot \| w_p(\xi_0) \| > e^{-\frac{\beta}{2}p} \| w_p(\xi_0) \|^{\frac{1}{2}} > e^{-\frac{\beta}{2}p}e^{\frac{cp}{2}} > e^{\frac{cp}{3}}$. $\square$

In analogy with Definition 3.3, we define for $\xi_0(s) \in \gamma$ the notion of a *fold period with respect to* $z_0$. This is the number $\ell$ such that $(Kb)^{\frac{\ell}{2}} \approx s$. If $\tau_0(\xi_0)$, the unit tangent vector to $\gamma$ at $\xi_0$, is split according to this definition, then the rejoining of the $E_i$-vector for $\ell < i < p$ has negligible effect. We may assume also that as we iterate, the sub-segment of $\gamma$ bound to $z_0$ acquires the same fold periods as $z_i$, and think of these as *secondary fold periods* for $\xi_i$ .

**Corollary 4.3** *Let the assumptions and notation be as in Corollary 4.2. We let* $p = p(\xi_0)$ *where* $|\xi_0 - z_0| = e^{-h}$ *and assume that* $z_p$ *is not in a fold period. Then*

*(a) the subsegment of* $T^p\gamma$ *between* $\xi_p$ *and* $z_p$ *contains a curve* $\geq e^{-K\beta h}$ *in length and with b-horizontal tangent vectors;*

*(b)*

$$\|\tau_p(\xi_0)\| \geq K^{-1}e^{h(1-\beta K)}.$$

**Proof:** (a) By definition, $|\xi_p - z_p| > e^{-\beta p}$. The part of $T^p\gamma$ in a fold period with respect to $z_0$ has length $\leq (Kb)^{\frac{p}{2}} \|DT\|^p$, and the rest have $b$-horizontal tangent vectors. To convert these estimates in $p$ into bounds involving $h$, use Corollary 4.2(a).

(b) Splitting $\tau_0$ using $e_p$, we see that $\|w_p\| \sim e^h\|\tau_p\|$. Combining this with Lemmas 4.11 and 4.9, we have $e^h\|\tau_p(\xi_0)\| \sim \|w_p(\xi_0)\| \approx \|w_p(z_0)\| > K^{-1}|\xi_p - z_p|e^{2h} \geq K^{-1}e^{-K\beta h}e^{2h}$. □

# 5 Pushing the Induction Forward

The goal of this section is to define $\Delta_{3N}$ and to prove that (IA1)–(IA6) hold up to time $3N$ for parameters in $\Delta_{3N}$. The key to this inductive step is the correct splitting of the $w_i^*$-vectors at free returns (Proposition 5.2). This is proved with the aid of another important fact, namely the control of points in $\partial R_k$ (Proposition 5.1).

## 5.1 Control of $\partial R_k$, $k \leq \theta N$

For $z \in \partial R_k$, let $\tau(z)$ denote a unit tangent vector to $\partial R_k$ at $z$.

**Proposition 5.1** *For every* $\xi_0 \in \partial R_0$ *and every* $k \leq \theta N$, $(\xi_0, \tau_0)$ *with* $\tau_0 = \tau(\xi_0)$ *is controlled up to time* $k$ *by* $\Gamma_k$.

**Proof:** The proof proceeds by induction. The correctness of splitting of $\tau_0$ is evident. We assume all $(\xi_0, \tau_0)$ have been controlled up to time $k-1$, so that it makes sense to speak of $\xi_k$ as being in a bound or free state. Suppose $\xi_k$ is bound to $z_i$ for some $z_0 \in \Gamma_{k-1}$. Since $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$, we have $z_i \in \mathcal{C}^{(j)} \setminus \mathcal{C}^{(j+1)}$ for some $j << i \leq k$. By Lemma 4.2, $\xi_k$ is h-related to $\Gamma_k$, and by Lemma 4.5, $\tau_k^*$ splits correctly, proving control at step $k$. Before proceeding to the free case, we state a lemma of independent interest:

**Lemma 5.1** *Let* $\gamma$ *be a subsegment of* $\partial R_k$. *If all the points on* $\gamma$ *are free, then* $\gamma$ *is a* $C^2(b)$*-curve.*

**Proof:** That $\tau_k$ is a $b$-horizontal vector is an immediate consequence of the splitting algorithm. As for curvature, we appeal to Lemma 2.4 after using Lemma 4.8 to establish that $\|\tau_k\| > K^{-K\theta(k-i)}\|\tau_i\|$ for all $i < k$. $\diamondsuit$

Returning to the proof of Proposition 5.1, let $\xi_k$ be a free return, and let $\gamma$ be the maximal free subsegment of $\partial R_k$ containing $\xi_k$. Since the end points of $\gamma$ are in bound state, they cannot be in $\mathcal{C}^{(k-1)}$ as explained earlier. This leaves two possibilities for the relation between $\gamma$ and $\mathcal{C}^{(k-1)}$.

*Case 1.* $\gamma$ passes through the entire length of a component of $\mathcal{C}^{(k-1)}$. In this case we know from (IA1) that there is a critical point $z_0 \in \gamma$. To see that every $\xi' \in \gamma \cap \mathcal{C}^{(0)}$ is h-related to $\Gamma_k$, start from $z_0$ and move away from it along $\gamma$. Using the $C^2(b)$ property of $\gamma$, the structure of critical regions (see (IA1)) and the fact that $\gamma \cap \partial R_i = \emptyset \ \forall i < k$, we observe that after leaving $\partial Q^{(k)}(z_0)$ one gets into $Q^{(k-1)}(z_0)$, then $Q^{(k-2)}(z_0)$, and so on, with $d_{\mathcal{C}}(\xi') \geq \rho^i$ for $\xi' \in Q^{(i-1)}(z_0) \setminus Q^{(i)}(z_0)$. For the splitting of $\tau(\xi')$, it follows from Lemma 4.1 and the $C^2(b)$ property of $\gamma$ that for $\xi' \in \gamma \cap Q^{(i-1)}(z_0) \setminus Q^{(i)}(z_0)$, $\angle(\tau(\xi'), \tau(\phi(\xi')) \leq \angle(\tau(\xi'), \tau(z_0)) + \angle(\tau(z_0), \tau(\phi(\xi')) < (Kb)|\xi' - z_0| + (Kb)^{\frac{i-1}{4}} < \varepsilon_0 d_{\mathcal{C}}(\xi')$.

*Case 2.* $\gamma$ does not intersect $\mathcal{C}^{(k-1)}$. Let $j < k$ be the largest integer such that $\gamma \cap \mathcal{C}^{(j-1)} \neq \emptyset$. Then there exists $z \in \gamma \cap (\hat{Q}^{(j)} \setminus Q^{(j)})$ for some $Q^{(j)}$. Suppose for definiteness that $z$ lies in the right component of $\hat{Q}^{(j)} \setminus Q^{(j)}$. Moving left along $\gamma$ from $z$, we note that since $\gamma \cap Q^{(j)} = \emptyset$, the left end point $\hat{z}$ of $\gamma$ must also be in the same component of $\hat{Q}^{(j)} \setminus Q^{(j)}$. H-relatedness and correct splitting are now proved as in Case 1 with $\hat{z}$ playing the role of $z_0$. We know $\tau(\hat{z})$ splits correctly because $\hat{z}$ is, by definition, in a bound state. $\square$

## 5.2 Extending control of $\Gamma_{\theta N}$-orbits to time $3N$

We continue to assume (IA1)–(IA6), which guarantee that if $w_0 = \binom{0}{1}$, then for all $z_0 \in \Gamma_{\theta N}$, $(z_0, w_0)$ is controlled up to time $N$ by $\Gamma_{\theta N}$. The next proposition plays a key role in the inductive process.

**Proposition 5.2** *If $z_0 \in \Gamma_{\theta N}$ satisfies $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $i \leq 3N$, then $(z_0, w_0)$ is automatically controlled by $\Gamma_{\theta N}$ up to time $3N$.*

**Proof:** From the condition that $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$, we have that $z_0$ is h-related to $\Gamma_{\theta N}$ up to time $3N$ (see the remark following (IA2) in Sect. 3.3.2), and that $p < K\alpha 3N << \frac{2}{3}N$. It suffices therefore to prove the correct splitting property at free returns. Proceeding inductively, we assume that $(z_0, w_0)$ is controlled up to time $k-1$ for some $k$ with $N \leq k \leq 3N$, and let $z_k$ be a free return. Then either $z_k \in \hat{Q}^{(j)} \setminus Q^{(j)}$ for some $j \leq \theta N$, or $z_k \in \mathcal{C}^{([\theta N])}$. In the latter case we let $j = [\theta N]$ for purposes of the following arguments.

**Claim 5.1** *There exists $j'$, $\frac{1}{3}j \le j' < j$, such that if*

$$\xi_0 = z_{k-j'} \qquad \text{and} \qquad u_0 = \frac{w_{k-j'}(z_0)}{\|w_{k-j'}(z_0)\|} \ ,$$

*then for* $0 \le s < j'$,

$$\|DT^s(\xi_0)u_0\| \ge \|DT\|^{-s}.$$

*Proof of Claim 5.1:* We consider the graph $\mathcal{G}$ of $i \mapsto \log\|w_i(z_0)\|$ for $k-j < i \le k$. Let $L$ be the (infinite) line through $(k,\ \log\|w_k\|)$ with slope $\log\|DT\|$. Then clearly, all the points in $\mathcal{G}$ lie above $L$. Let $P$ be the intersection of $L$ with the line $x = k - \frac{1}{3}j$. We let $L$ be pivoted at $P$ and rotate it clockwise until it hits some point in $\mathcal{G}$. (Draw a picture!) Let $k-j'$ be the first coordinate of the first point hit. Then $\frac{1}{3}j \le j' < j$, and since all points in $\mathcal{G}$ lie above $L$, Claim 5.1 is proved if we can show that in its final position, the slope of $L$ is $\ge -\log\|DT\|$. This is true because $z_k$ being free, there must be some $j''$ with $\frac{2}{3}j \le j'' < j$ such that $z_{k-j''}$ is not in a fold period, otherwise the bound period initiated at the same time as this (very long) fold period would last beyond $z_k$. By Lemma 4.8, $\|w_{k-j''}\| < \|w_k\|$. Thus one cannot rotate $L$ to a slope $< -\log\|DT\|$ without first hitting the point $(k-j'',\ \log\|w_{k-j''}\|) \in \mathcal{G}$. $\diamondsuit$

Now by Lemma 2.3, there exists an integral curve $\gamma$ of the most contracted field of order $j'$ through $\xi_0$ having length $\mathcal{O}(1)$. Since $\gamma$ follows roughly the direction of $e_1$, it has slope $> K^{-1}\delta$ outside of $\mathcal{C}^{(0)}$ and is roughly a parabola inside $\mathcal{C}^{(0)}$ (Lemma 2.9). In both cases, $\gamma$ meets $\partial R_0$. Let $\xi_0' \in \gamma \cap \partial R_0$. Then

$$|\xi_s - \xi_s'| < (K^2 b)^s$$

for all $0 \le s \le j'$. Our next claim is made possible by Proposition 5.1.

**Claim 5.2** *$\xi_{j'}'$ is a free return.*

*Proof of Claim 5.2:* If not, then $\xi_{j'}'$ would be bound to $\hat{z}$, a point on a critical orbit, and we would have $\xi_{j'}, \xi_{j'}' \in \hat{Q}^{(i)}(\hat{z})$ for some $i << j' < j$ with $d_{\mathcal{C}}(\xi_{j'}) \approx d_{\mathcal{C}}(\xi_{j'}') \approx d_{\mathcal{C}}(\hat{z}) > e^{-\alpha j'}$. This contradicts our assumption that $\xi_{j'} = z_k$ is in $\hat{Q}^{(j)}$ or in $\mathcal{C}^{([\theta N])}$, for in either case, $d_{\mathcal{C}}(z_k) < \rho^{j-1}$. $\diamondsuit$

**Claim 5.3** *With $u_0$ as in Claim 5.1, let*

$$\tau_i = DT^i(\xi_0')\tau_0, \qquad u_i = DT^i(\xi_0)u_0,$$

*and let $\theta_i$ be the angle between $u_i$ and $\tau_i$. Then $\theta_{j'} \le b^{\frac{j'}{2}}$.*

*Proof of Claim 5.3:* Write $A = DT(\xi_{i-1})$ and $A' = DT(\xi_{i-1}')$. Then

$$\begin{aligned} \theta_i = \frac{\|\tau_i \times u_i\|}{\|\tau_i\| \cdot \|u_i\|} \quad &= \quad \frac{1}{\|\tau_i\| \cdot \|u_i\|}\|A'\tau_{i-1} \times A'u_{i-1} + A'\tau_{i-1} \times (A - A')u_{i-1}\| \\ &\le \quad \frac{\|\tau_{i-1}\|}{\|\tau_i\|} \cdot \frac{\|u_{i-1}\|}{\|u_i\|} \cdot (|\det(A')|\theta_{i-1} + K|\xi_i - \xi_i'|) \\ &\le \quad \frac{\|\tau_{i-1}\|}{\|\tau_i\|} \cdot \frac{\|u_{i-1}\|}{\|u_i\|} \cdot (b\theta_{i-1} + K(K^2 b)^{i-1}). \end{aligned}$$

Applying this relation for $\theta_i$ recursively, we obtain

$$\theta_{j'} < \left( \sum_{i=0}^{j'} \frac{\|\tau_i\|}{\|\tau_{j'}\|} \cdot \frac{\|u_i\|}{\|u_{j'}\|} \right) (K^2 b)^{j'-1}.$$

Since both $z_k$ and $\xi_{j'}$ are free returns, we may use Lemma 4.8 to bound the sum in brackets by $\sum_i K^{4\theta(j'-i)} < 2K^{4\theta j'}$, completing the proof of Claim 5.3. $\diamondsuit$

We are finally ready to prove that $w_k(z_0)$ splits correctly. Recall that $\xi_{j'} = z_k \in \hat{Q}^{(j)}$ or $Q^{([\theta N])}$. Since $|\xi_{j'} - \xi'_{j'}| < (K^2 b)^{j'}$, $\xi'_{j'} \in \partial R_{j'}$ and $j' < j$, we have $\xi'_{j'} \in \partial Q^{(j')}(z_k)$. By our inductive hypothesis, $\tau_{j'}(\xi'_0)$ splits correctly. Since $\angle(w_k(z_0), \tau(\xi'_{j'})) \leq b^{\frac{j'}{2}}$ (Claim 5.3), $\angle(\tau(\phi(\xi'_{j'})), \tau(\phi(z_k))) = \mathcal{O}(b^{\frac{j'}{4}})$ and $|d_{\mathcal{C}}(\xi'_{j'}) - d_{\mathcal{C}}(z_k)| = \mathcal{O}(b^{\frac{j'}{4}})$ (Lemma 4.1), it suffices to prove that $b^{\frac{j'}{4}} << d_{\mathcal{C}}(z_k)^2$. In the case where $z_k \in \hat{Q}^{(j)} \setminus Q^{(j)}$, this is trivial as $d_{\mathcal{C}}(z_k) \sim \rho^j$. In the case where $z_k \in Q^{([\theta N])}$, since $d_{\mathcal{C}}(z_k) > e^{-\alpha k}$, we have $d_{\mathcal{C}}(z_k)^2 > e^{-6\alpha N}$, which we may assume is $>> b^{\frac{1}{12}\theta N} \geq b^{\frac{1}{4}j'}$. This completes the proof of Proposition 5.2. $\square$

## 5.3 Verification of (IA1)–(IA6) up to time $3N$

**Step 1** *Deletion of parameters.* We delete from $\Delta_N$ all $(a, b)$ for which there exists $z_0 \in \Gamma_{\theta N}$ and $i$, $N < i \leq 3N$, such that

$$d_{\mathcal{C}}(z_i) < e^{-\alpha i} \qquad \text{or} \qquad \|w_i^*(z_0)\| < e^{ci}.$$

The set of remaining parameters is called $\Delta_{3N}$. We do not claim in (IA1)–(IA6) that $\Delta_{3N}$ has positive measure or even that it is nonempty; this is discussed in Section 6. Steps 2–5 below apply to $T = T_{a,b}$ for $(a, b) \in \Delta_{3N}$.

**Step 2** *Updating of* $\Gamma_{\theta N}$. For each $z_0 \in \Gamma_{\theta N}$, since $\|w_i\|$ grows exponentially (Step 1 and Lemma 4.6), there exists a unique $z'_0$ on the component of $\partial\mathcal{C}^{(k)}$ containing $z_0$ that is a critical point of order $3N$ (Lemma 2.10). Let $\Gamma'_{\theta N}$ be the set of these $z'_0$, i.e. $\Gamma'_{\theta N}$ is a copy of $\Gamma_{\theta N}$ updated to order $3N$.

**Step 3** *Construction of* $\Gamma_{3\theta N}$ *and* $\mathcal{C}^{(k)}$*,* $\theta N < k \leq 3\theta N$*.* We establish control of $\partial R_k$ as in Sect. 5.1, with one minor difference as explained in the next paragraph. Assuming that all has been accomplished for $k-1$. Then $R_k$ meets each component $Q^{(k-1)}$ of $\mathcal{C}^{(k-1)}$ in at most a finite number of strips bounded by free, and hence $C^2(b)$, curves. Let $\gamma$ be one of these curves. By Lemma 2.11, there exists a critical point $\hat{z}_0 \in \gamma$ of order $\hat{m} = min\{3N, -\log d(z_0, \gamma)^{\frac{1}{2}}\}$ where $z_0 \in \Gamma'_{\theta N}$ lies on the boundary of the component $Q^{([\theta N])}$ containing $\gamma$. Since $d(z_0, \gamma) = \mathcal{O}(b^{\frac{\theta N}{2}})$, we have, assuming $\theta$ is chosen with $e^{-3N} > K^{-N} > b^{\frac{\theta N}{4}}$, that $\hat{z}_0$ is of order $3N$.

To continue, we need to set bindings for points in $\partial R_k$. Technically, only $z_0 \in \Gamma_{\theta N}$ (and not the critical points on $\partial R_i$, $\theta N < i \leq k$) can be used. This is of no concern

to us for the following reason: for $k'$ with $k < k' \le 3\theta N$, only those parts of $\partial R_{k'}$ that are free are involved in the construction of $\mathcal{C}^{(k')}$; and for $\xi_0 \in \partial R_k \cap \mathcal{C}^{([\theta n])}$, independent of which $z_0 \in Q^{([\theta n])}(\xi_0)$ we think of it as bound to, $\xi_i$ will remain in bound state through time $3\theta N$ because $|\xi_i - z_i| \le K^{3\theta N}\rho^{\theta N} << e^{-3\beta\theta N}$.

The newly constructed critical points in $\partial R_k$, $N < k \le 3N$, together with $\Gamma'_{\theta N}$ form $\Gamma_{3\theta N}$. We have completed the verification of (IA1) up to time $3N$.

**Step 4** *Updating the definitions of* $d_{\mathcal{C}}(\cdot)$ *and* $\phi(\cdot)$. Using $\Gamma_{3\theta N}$ and $\mathcal{C}^{(k)}$, $k \le [3\theta N]$, we reset these definitions for $z \in \mathcal{C}^{([\theta N])}$ in accordance with Definition 3.2. Since $|\text{old}\phi(z) - \text{new}\phi(z)| = \mathcal{O}(b^{\frac{\theta N}{4}})$ and $|\tau(\text{old}\phi(z)) - \tau(\text{new}\phi(z))| = \mathcal{O}(b^{\frac{\theta N}{4}})$ (Lemma 4.1), these changes have essentially no effect on the correctness of splitting for points with $d_{\mathcal{C}}(\cdot) > b^{\frac{3\theta N}{20}}$. The relations in (IA5) are also not affected.

**Step 5** *Verification of (IA2)–(IA6) for* $i \le 3N$. This is carried out in 3 stages.

(1) First we argue that for $z_0 \in \Gamma_{\theta N}$ (we really mean $\Gamma_{\theta N}$, not $\Gamma'_{\theta N}$), (IA2)–(IA6) hold for $i \le 3N$: (IA2) and (IA4) hold by design; (IA3) is given by Proposition 5.2, and (IA5) and (IA6) are proved in Sect. 4.3 with $m = 3N$.

(2) With the properties of $\Gamma_{\theta N}$ in (1) having been established, we observe that continuing to use $\Gamma_{\theta N}$ as the source of control, the material in Sects. 4.2 and 4.3 are now valid for times up to $min(m, 3N)$.

(3) We are now ready to argue that (IA2)–(IA6) hold for all $z'_0 \in \Gamma_{3\theta N}$. For each $z'_0 \in \Gamma_{3\theta N}$, whether it is in $\Gamma'_{\theta N}$ or of generation $> \theta N$, there exists $z_0 \in \Gamma_{\theta N}$ such that $|z'_0 - z_0| = \mathcal{O}(b^{\frac{\theta N}{4}})$. This implies, for $i \le 3N$, that $|z'_i - z_i| < b^{\frac{\theta N}{4}}\|DT\|^{3N} << e^{-\beta 3N}$ provided $\theta$ is chosen so that $b^{\frac{\theta}{4}}\|DT\|^3 < \frac{1}{2}e^{-\beta}$. (IA2) follows immediately from the corresponding condition for $z_0$. Regarding $z'_0$ as bound to $z_0$ for at least $3N$ iterates, (IA3) and (IA4) follow from property (IA6) of $z_0$. Finally, regarding $(z'_0, \binom{0}{1})$ as controlled by $\Gamma_{\theta N}$ up to time $3N$, we obtain (IA5) and (IA6) from Lemmas 4.9-4.11 and Corollary 4.2.

**Conclusions from Sections 3–5:** After letting $N$ go to infinity, we have defined for each $T = T_{a,b}$ with $(a, b) \in \Delta := \cap_N \Delta_N$ a set $\mathcal{C}$ given by $\mathcal{C} = \cap_{i \le 0}\mathcal{C}^{(i)}$. This is the *critical set* in Theorem 1. Let $\Gamma$ be the set to which $\Gamma_{\theta N}$ converges as $N \to \infty$. An equivalent characterization of $\mathcal{C}$ is that it is the set of accumulation points of $\Gamma$. Clearly, the properties that $d_{\mathcal{C}}(z_i) \ge e^{-\alpha i}$ and $\|w_i\|$ grows exponentially are passed on to points in $\mathcal{C}$. We have thus completed the proof of Theorem 1 modulo the positivity of the measure of $\Delta$.

# 6 Measure of Selected Parameters

In this section we fix $b > 0$ and consider the 1-parameter family $a \mapsto T_{a,b}$. Let $\Delta_b = \{a : (a,b) \in \Delta\}$. The Lebesgue measure of a set $A \subset \mathbb{R}$ is denoted by $|A|$. More generally, we use $|\cdot|$ to denote the measure on curves induced by arc length. The purpose of this section is to prove that $|\Delta_b| > 0$ for all sufficiently small $b > 0$.

## 6.1 Phase-space dynamics and curves of critical orbits

Assuming $\delta = e^{-\mu^*}$ for some $\mu^* \in \mathbb{Z}^+$, let $\mathcal{P} = \{I_{\mu j}\}$ be the partition of the interval $(-\delta, \delta)$ defined as follows: for $\mu \geq \mu^*$, let $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$, and let each $I_\mu$ be further subdivided into $\mu^2$ subintervals of equal length called $I_{\mu j},\ j = 1, 2, \cdots \mu^2$; for $\mu \leq -\mu^*$, let $I_{\mu j} = -I_{(-\mu)j}$.

Next let $\gamma$ be a curve with nearly horizontal tangent vectors. We assume for simplicity that $\gamma$ meets only one component $Q^{(0)}$ of $\mathcal{C}^{(0)}$, and let $\hat{z} = (\hat{x}, \hat{y})$ be a point near the center of $Q^{(0)}$. The partition $\mathcal{P}_{\gamma,\hat{z}}$ on $\gamma$ is defined to be $(\psi^{-1}\mathcal{P})|\gamma \cup \{I^\pm\}$ where $\psi(x, y) = x - \hat{x}$ and $I^\pm$ are the two components of $\gamma \setminus \psi^{-1}(-\delta, \delta)$. An element of $\mathcal{P}_{\gamma,\hat{z}}$ is said to have "full length" if its image under $\psi$ is either equal to some $I_{\mu j}$ or longer than all the $I_{\mu j}$'s. When $\gamma$ and $\hat{z}$ are understood, we often refer to $\mathcal{P}_{\gamma,\hat{z}}$ simply as $\mathcal{P}$ and $(\psi^{-1}I_{\mu j}) \cap \gamma$ as $I_{\mu j}$.

Before proceeding to the estimation of $|\Delta_b|$, we consider first the following problem in phase-space dynamics. The estimation of $|\Delta_b|$ includes an argument parallel to and more complicated than this.

#### A model problem in phase-space dynamics

Let $T = T_{a,b}$ with $(a, b) \in \Delta$. Recall from the proof of Proposition 5.1 that if $\gamma \subset \partial R_k$ is a maximal free segment meeting some $Q^{(0)}$, then either $\gamma \cap Q^{(0)}$ contains a critical point $\hat{z} \in \Gamma$ or the entire segment $\gamma \cap Q^{(0)}$ is h-related to some $\hat{z} \in \Gamma$. In both cases, $\mathcal{P}_{\gamma,\hat{z}}$ is the partition of choice on $\gamma$. Note that for $z \in I_{\mu j},\ d_{\mathcal{C}}(z) \approx e^{-|\mu|}$.

Let $\omega_0$ be a subsegment of $\partial R_0$, and write $\omega_i := T^i\omega_0$. We assume that (i) for all $z_0 \in \omega_0,\ d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $i \leq N$, (ii) each $\omega_i,\ i < N$, is contained in three consecutive $I_{\mu j}$, and (iii) $\omega_N$ is free and is approximately equal to some $I_{\mu_0 j_0}$. The problem is to find a lower estimate for the measure of $\{z_0 \in \omega_0 : d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $i\}$.

We may assume that all the points in $\omega_N$ have the same bound period, and let $i_1 > N$ be the first moment in time after the expiration of this bound period when $\omega_{i_1} \cap \mathcal{C}^{(0)}$ contains an $I_{\mu j}$ of full length. This must happen at some point, for the length of $\omega_i$ grows by a factor $> K$ between successive free returns (Corollary 4.3). It is easy to check that $d_{\mathcal{C}} > e^{-\alpha i}$ is not violated between times $N$ and $i_1$. Let $\{\omega\}$ be the partition $\mathcal{P}$ on $\omega_{i_1}$ with end segments attached to their neighbors if they are not of full length. We delete those $\omega$'s that contain some $z$ with $d_{\mathcal{C}}(z) < e^{-\alpha i_1}$. For each $\omega$ that is kept, we repeat the procedure above with $\omega$ in the place of $\omega_N$, that

is, we iterate until $\omega$ makes a free return at time $i_2 = i_2(\omega)$ with $T^{i_2-i_1}\omega$ containing an $I_{\mu j}$ of full length. We then partition $T^{i_2-i_1}\omega$, discard subsegments that violate $d_{\mathcal{C}} > e^{-\alpha i_2}$, and continue to iterate the rest.

We estimate the fraction of $\omega_{i_1}$ deleted at time $i_1$ as follows. Since $\omega_N \approx I_{\mu_0 j_0}$, the bound period $p$ is $\leq K|\mu_0|$. From Corollary 4.3, we see that $\omega_{i_1}$ has length $> \frac{K^{-1}}{\mu_0^2} e^{-\beta K|\mu_0|} > e^{-2\beta|\mu_0|K}$. Now $|\mu_0| \leq \alpha N$ and $i_1 > N + p_0$ where $p_0 > 0$ is a lower bound for all bound periods. Then

$$\frac{|\{z \in \omega_{i_1} : d_{\mathcal{C}}(z) < e^{-\alpha i_1}\}|}{|\omega_{i_1}|} \; < \; \frac{2e^{-\alpha(N+p_0)}}{e^{-2K\alpha\beta N}} \; < \; e^{-\frac{1}{2}\alpha N}$$

assuming $N$ is sufficiently large. Similarly, for each subsegment $\omega \approx I_{\mu j}$ of $\omega_{i_1}$ that is kept, the fraction of $T^{i_2-i_1}\omega$ deleted at time $i_2$ is $< e^{-\frac{1}{2}\alpha i_1} < e^{-\frac{1}{2}\alpha(N+p_0)}$, and so on. To estimate the total measure of $\omega_0$ deleted, these fractions have to be pulled back to $\omega_0$. This involves a *distortion estimate* for $DT^i$ along certain subsegments of $\partial R_k$. Using the fact that this distortion is uniformly bounded (Lemma 8.2), we see that the fraction of $\omega_0$ deleted in this procedure is $< K \sum_i e^{-\frac{1}{2}\alpha(N+ip_0)} < Ke^{-\frac{1}{2}\alpha N}$.

We remark that the scheme in this paragraph relies on the fact that $\omega_N$ has a certain minimum length depending on $N$, otherwise the entire segment may be obliterated before time $i_1$ is reached.

**Strategy for estimating $|\Delta_b|$**

Since $b$ is fixed throughout this discussion, let us for notational simplicity omit mention of it and write $\Delta, \Delta_N$ and $T_a$ instead of $\Delta_b, \Delta_b \cap \Delta_N$ and $T_{a,b}$. Let $N$ be fixed. The problem is to estimate the measure of parameters deleted between times $N$ and $3N$. Our strategy is as follows: For $\hat{a} \in \Delta_N$ and $z_0 \in \Gamma_{\theta N}(\hat{a})$, let $a \mapsto z_0(a)$ be defined on an interval containing $\hat{a}$. We consider

$$\gamma_0 \;\to\; \gamma_1 \;\to \gamma_2 \;\to\; \cdots \qquad\qquad \text{where} \qquad\qquad \gamma_i(a) := z_i(a) = T_a^i(z_0(a)),$$

and estimate the measure of the set of $a$ for which $z_i(a)$ violates (IA2) or (IA4).

The idea behind this line of proof is that *qualitatively, the evolution of* $\gamma_0$ *is similar to that of* $\omega_0$ in the model phase-space problem. If this is true, then the measure deleted on account of (IA2) can be estimated analogously. To understand why the $\gamma_i$'s behave like phase curves, i.e. curves that are obtained through the iteration of $T_a$, observe the way in which $\frac{d}{da}\gamma_i$, the tangent vector to the curve $a \mapsto \gamma_i(a)$, is transformed: if $\|\frac{d}{da}\gamma_i(a)\| >> 1$, then $\frac{d}{da}\gamma_{i+1}(a) \approx DT_a(\gamma_i(a))\frac{d}{da}\gamma_i(a)$; that is to say, $\gamma_{i+1} \approx T_a \circ \gamma_i$ near $\gamma_i(a)$.

**Issues to be addressed**

1. *Similarity of space- and a-derivatives.* This is the first and most important step in justifying the thinking in the last paragraph. Let $\gamma_0$ be as above. In Sect. 6.2, we

show that $\frac{d}{da}\gamma_i \sim DT^i\binom{0}{1}$. As we will see, this is made possible by our transversality condition on $\{f_a\}$ in Sect. 1.1. The only other prerequisite for this comparison is that the slopes of $\gamma_0$ be suitably bounded. This is verified in Sect. 6.3 for curves corresponding to critical points of all generations and all orders.

2. *Dynamics of the curves $a \mapsto \gamma_i(a)$.* Our next step is to show that *as curves parametrized by $a$*, the $\gamma_i$ have properties similar to those of $\omega_i$. For example, with $\Gamma_{\theta N}$ moving with $a$, how is $d_{\mathcal{C}}(z_i(a))$ affected? Other properties include the geometry of free segments, quadratic behavior of the type in Sect. 4.3, distortion estimates along $\gamma_i$ etc. These questions are discussed in Sect. 6.4.

3. *Deletions of parameters in violation of (IA2) or (IA4).* We consider $z_0 \in \Gamma_{\theta N}$ one at a time, and let $\gamma_0$ be the corresponding curve of critical points. Assuming the success of the last step, deletions on $\gamma_0$ on account of (IA2) are estimated following the scheme outlined in the model problem. Estimates for the measure of parameters deleted on account of (IA4) are discussed in Sect. 6.5.

4. *Combined effect of deletions corresponding to all $z_0 \in \Gamma_{\theta N}$.* Obviously, we need to multiply the measure of the parameters deleted on each $\gamma_0$ by the cardinality of $\Gamma_{\theta N}$, but there are technical considerations: As in our phase-space model, to get started we need $\gamma_N$ to have a certain minimum length. This raises the question of the length of the parameter interval on which each $a \mapsto z_0(a)$ can be continued (this problem appears already in Sect. 6.3). Also relevant is the combined effect of deletions on all critical curves prior to time $N$. The final estimate is made in Sect. 6.6.

The idea to relate parameter-space dynamics to phase-space dynamics is, of course, not new. Two results on 1-dimensional maps are cited without proof and used in this section: a transversality condition from [TTY] is used in Sect. 6.2 and a large deviation estimate from [BC2] is used in Sect. 6.5.

## 6.2 Equivalence of space- and $a$-derivatives

The setting of this subsection is as follows: For fixed $b > 0$, let $\hat{a} \in \Delta_N$ for some $N$, and let $z_0 = z_0(\hat{a}) \in \Gamma_{\theta N}(\hat{a})$. Let $n \le N$. Then $z_0$ obeys (IA2) and (IA4) and the conclusions of Lemmas 4.6–4.8 up to time $n$. We assume also that $z_0(\hat{a})$ has a smooth continuation $a \mapsto z_0(a)$ to an $a$-interval containing $\hat{a}$. Let $w_i = DT^i_{\hat{a}}(z_0(\hat{a}))\binom{0}{1}$ and $\tau_i = \frac{dz_i}{da}(\hat{a})$. The goal of this subsection is to compare $w_i$ and $\tau_i$. Let $\tau_0 = (\tau_{0,1}, \tau_{0,2})$.

**Proposition 6.1** *Given $\bar{\tau} > 0$, there exist constants $\lambda_2 > \lambda_1 > 0$ and a small $\varepsilon > 0$ such that the following holds: If $(\hat{a}, b)$ is sufficiently near $(a^*, 0)$, $z_0(\hat{a})$ is as above, $\|\tau_0\| < \bar{\tau}$ and $|\tau_{0,2}| < \varepsilon$, then for all $i \le n$,*

$$\lambda_1 \le \frac{\|\tau_i\|}{\|w_i\|} \le \lambda_2.$$

We will show below that once we have $\|\tau_i\| \sim \|w_i\|$ for some $i$ with $\|w_i\|$ sufficiently large, then this relationship will hold from there on. The estimate for the initial stretch is guaranteed by our transversality condition on the 1-dimensional family $\{f_a\}$. We recall a relevant result from 1-dimension:

Let $f$ and $\{f_a\}$ be as in Sect. 1.1. Let $x_0$ be a critical point of $f$, and let $p = f(x_0)$. Since $f = f_{a^*}$, we write $x_0(a^*) = x_0,\ p(a^*) = p$, and let $a \mapsto x_0(a)$ and $a \mapsto p(a)$ be the continuation of $x_0$ and $p$ as defined in Sect. 1.1. Let $x_k(a) = f_a^k(x_0(a))$. We will use $(\cdot)'$ to denote differentiation with respect to $x$.

**Lemma 6.1** ([TTY], Proposition VII.7) *As $k \to \infty$,*

$$Q_k(a^*) \ := \ \frac{\frac{dx_k}{da}(a^*)}{(f^{k-1})'(x_1(a^*))} \quad \to \quad \lambda_0 := \ \frac{dx_1}{da}(a^*) - \frac{dp}{da}(a^*).$$

The transversality condition in Sect. 1.1, Step II, states that $\lambda_0 \neq 0$. We will also need the following technical lemma the proof of which is given in Appendix B.9.

**Lemma 6.2** *There exist constants $K$ and $c' > 0$ such that for every $0 \le s < i$, we have*

$$\|DT^{i-s}(z_s)\| \le Ke^{-c's}\|w_i\|.$$

**Proof of Proposition 6.1:** Since

$$\tau_i = DT(z_{i-1})\tau_{i-1} + \psi(z_{i-1})$$

where $\psi(z) = \frac{\partial(T_a z)}{\partial a}(\hat{a})$, we have inductively

$$\tau_i = DT^i(z_0)\tau_0 + \sum_{s=1}^{i} DT^{i-s}(z_s)\psi(z_{s-1}).$$

The upper estimate for $\frac{\|\tau_i\|}{\|w_i\|}$ follows from Lemma 6.2 and the uniform boundedness of $\|\psi(\cdot)\|$:

$$\begin{aligned}\frac{\|\tau_i\|}{\|w_i\|} &\le \frac{\|DT^i(z_0)\tau_0\|}{\|w_i\|} + \sum_{s=1}^{i} \frac{\|DT^{i-s}(z_s)\psi(z_{s-1})\|}{\|w_i\|} \\ &< K\|\tau_0\| + K\sum_{s=1}^{\infty} e^{-c's} := \lambda_2.\end{aligned}$$

To obtain a lower bound for $\frac{\|\tau_i\|}{\|w_i\|}$, we pick $k_0$ large enough that $|Q_{k_0}(a^*)| > \frac{1}{2}|\lambda_0|$ where $Q_{k_0}$ and $\lambda_0$ are as in Lemma 6.1, and decompose $\tau_i$ into $\tau_i = I + II$ where

$$I = DT^i(z_0)\tau_0 + \sum_{s=1}^{k_0} DT^{i-s}(z_s)\psi(z_{s-1}),$$

$$II = \sum_{s=k_0+1}^{i} DT^{i-s}(z_s)\psi(z_{s-1}).$$

Again by Lemma 6.2, we have

$$\frac{\|II\|}{\|w_i\|} < \sum_{s=k_0+1}^{\infty} Ke^{-c's}.$$

We will show $\frac{\|I\|}{\|w_i\|} > K_0^{-1}|\lambda_0|$ for some $K_0$, and assume $k_0$ is chosen so that $\sum_{s>k_0} Ke^{-c's}$ $<< K_0^{-1}|\lambda_0|$. Write

$$I = DT^{i-k_0}(z_{k_0})V$$

where

$$V = DT^{k_0}(z_0)\tau_0 + \sum_{s=1}^{k_0} DT^{k_0-s}(z_s)\psi(z_{s-1}).$$

**Claim 6.1**

$$\|V\| \ > \ \frac{1}{3}\ \frac{\|w_{k_0}\|}{\|w_1\|}\ |\lambda_0|,$$

*and the second component of* $V$ *tends to* $0$ *as* $(\hat{a}, b) \to (a^*, 0)$.

*Proof of Claim 6.1:* Let $z_0 \to (x_0, 0)$ as $(\hat{a}, b) \to (a^*, 0)$. The two terms of $V$ are estimated as follows:

(i) $\|DT^{k_0}(z_0)\tau_0\| < K|\tau_{0,2}|$ for $(\hat{a}, b)$ sufficiently near $(a^*, 0)$. This is because $k_0$ is a system constant, and writing $T^{k_0}_{a^*,0} = (T^1, T^2)$, we have

$$DT^{k_0}_{\hat{a},b}(z_0)\tau_0 \ \to \ \left(\frac{\partial T^1}{\partial x}(x_0, 0)\tau_{0,1} + \frac{\partial T^1}{\partial y}(x_0, 0)\tau_{0,2},\ 0\right) = \left(\frac{\partial T^1}{\partial y}(x_0, 0)\tau_{0,2},\ 0\right).$$

(ii) For $(\hat{a}, b)$ sufficiently near $(a^*, 0)$, $z_s$ stays out of $\mathcal{C}^{(0)}$ for $> k_0$ iterates, and

$$\frac{\sum_{s=1}^{k_0} DT^{k_0-s}(z_s)\psi(z_{s-1})}{\|w_{k_0}\|/\|w_1\|} \ \to \ \left(\frac{\sum_{s=1}^{k_0}(f^{k_0-s})'(x_s(a^*))\frac{d}{da}(f_a(x_{s-1}))(a^*)}{\pm(f^{k_0-1})'(x_1(a^*))},\ 0\right)$$

$$= \left(\pm\sum_{s=1}^{k_0}\frac{\frac{d}{da}(f_a(x_{s-1}))(a^*)}{(f^{s-1})'(x_1(a^*))},\ 0\right),$$

which by a simple computation is equal to $(\pm Q_{k_0}(a^*),\ 0)$. ◊

Assuming further that the $x$-coordinate of $z_{k_0}$ is in a small neighborhood of $x_{k_0}$ (which is bounded away from the critical set), and that $z_s$ stays outside of $\mathcal{C}^{(0)}$ for

$>> k_0$ iterates, we have that the slope of $e_{i-s}(z_s)$ is bounded below by some $K^{-1}$. This together with Claim 6.1 gives

$$\|DT^{i-k_0}(z_{k_0})V\| > K^{-1}\|DT^{i-k_0}(z_{k_0})w_{k_0}\| \ \frac{\|V\|}{\|w_{k_0}\|} > K_0^{-1}\|w_i\| \ |\lambda_0|.$$

□

We will also need an estimate on the angle between $\tau_i$ and $w_i$, which we denote by $\theta_i$. The assumptions are as in Proposition 6.1 .

**Lemma 6.3** *If $z_i$ is a free return, then $\theta_i < \frac{K}{\|\tau_i\|}$.*

**Proof:**

$$| \sin\theta_i \, | \leq \ \frac{1}{\|\tau_i\|}\left(\sum_{s=1}^{i}\frac{1}{\|w_i\|}\|w_i \times DT^{i-s}(z_s)\psi(z_{s-1})\| + \frac{\|w_i \times DT^i(z_0)\tau_0\|}{\|w_i\|}\right)$$

$$\leq \frac{1}{\|\tau_i\|}\left(\sum_{s=1}^{i}\frac{\|w_s\|}{\|w_i\|}\ \left\|\frac{w_s}{\|w_s\|} \times \psi(z_{s-1})\right\| \ b^{i-s} + \frac{\|\tau_0\|}{\|w_i\|}\ b^i\right) \leq \frac{K}{\|\tau_i\|}\sum_{s=0}^{\infty} b^s.$$

The last inequality is valid if, for example, $\|w_s\| \leq \|w_i\|$ for all $s \leq i$, which is the case at free returns. □

## 6.3 Initial data for critical curves

The goal of this subsection is to verify the conditions on $\tau_0$ in Proposition 6.1 for critical curves of all generations and all orders. Our plan of proof is as follows:

1. We obtain information on the slopes of critical curves of generation $i$ by comparing them to critical curves of generation $i-1$. Following [BC2], this is done using a lemma of Hadamard, which requires that the intervals of definition of the critical curves be sufficiently long. We are thus led to the following question: on how long of a parameter interval can one continue a critical curve with reasonable properties?

2. As the order of a critical point tends to infinity, the length of the parameter interval on which it is defined goes to zero. This makes it necessary for us to prove our results in two steps, to first work with critical points having orders commensurate with their generations, and then to pass the bounds on to curves corresponding to higher orders.

### 6.3.1 Stability of critical regions

In Sections 3–5, we construct for $N = N_0, 3N_0, 3^2N_0, \cdots$ a parameter set $\Delta_N$ such that for $a \in \Delta_N$, $\Gamma_{\theta N}$ is well defined and consists of critical points of generation $\theta N$ and order $N$. Let us denote this set by $\Gamma_{\theta N,N}$. In the discussion to follow, it will

be convenient to consider $\Gamma_{i,n}$ for arbitrary $i \leq n$. We define these sets formally as follows:

First we fix $a \in \Delta_N$, and define $\Gamma_{i,N}$, $\theta N < i \leq N$, inductively by carrying out the steps in Section 5 in a slightly different order. Assuming that $\Gamma_{i-1,N}$ is defined and all the points in $\partial R_0$ are controlled for $i-1$ iterates, we define $\mathcal{C}^{(i)}$ and $\Gamma_{i,N}$. Immediately, we observe that the newly constructed critical points are controlled by $\Gamma_{\theta N,N}$. In particular, they satisfy (IA2) and (IA4) (with possibly slightly weaker constants) and can be used for binding. For free segments of $\partial R_i$ that lie in $\mathcal{C}^{(0)}$, we may then set binding as in the proof of Proposition 5.1, and proceed to step $i+1$.

For $n$ with $N < n \leq 3N$, let $\Delta_n := \{a \in \Delta_N$: (IA2) and (IA4) are satisfied up to time $n$ for orbits from $\Gamma_{\theta N}\}$. A slight extension of the argument above defines $\Gamma_{i,n}$ for all $a \in \Delta_n$ and $i \leq n$.

Finally, we introduce for each $n$ the parameter set $\tilde{\Delta}_n$, which has the same definition as $\Delta_n$ except that in the definition of $\tilde{\Delta}_N$, $N = N_0, 3N_0, \cdots$, (IA2) and (IA4) are replaced by $d_{\mathcal{C}}(z_j) > \frac{1}{2}e^{-\alpha j}$ and $\|w_j^*\| > \frac{1}{2}c_0 e^{cj}$. One checks easily that all the results in Sections 3–5 are valid under these slightly relaxed rules, as is the discussion in the last two paragraphs, so that $\Gamma_{i,n}$ is defined for all $a \in \tilde{\Delta}_n$ and $i \leq n$.

We remark before proceeding further that built into our definition of $\Gamma_{i,n}$ for $\frac{N}{3} < n < N$ is the property that $z_0 \in \Gamma_{i,n}$ has all the properties of $\tilde{z}_0 \in \Gamma_{\theta N,N}$ (except for the factor $\frac{1}{2}$) up to time $n$. In particular, Proposition 6.1 applies to $\hat{a} \in \tilde{\Delta}_n$ and $z_0 = z_0(\hat{a}) \in \Gamma_{i,n}$.

**Definition 6.1** *For $i \leq n$, an interval $J \subset \tilde{\Delta}_n$ and $\hat{a} \in J$, we say $\Gamma_{i,n}(\hat{a})$ has a* **smooth continuation** *to $J$ if there is a map $g : \Gamma_{i,n}(\hat{a}) \times J \to R_0$ such that*

*- $g(\cdot, a) = \Gamma_{i,n}(a)$ for all $a$ and*

*- for each $z \in \Gamma_{i,n}(\hat{a})$, $a \mapsto g(z,a)$ is smooth.*

*Likewise one has the notion of the* **critical regions $\mathcal{C}^{(i)}$ deforming continuously** *as $a$ ranges over $J$.*

**Lemma 6.4** *Let $\hat{a} \in \Delta_n$ and $J = [\hat{a} - \rho^{2n}, \hat{a} + \rho^{2n}]$. Then $J \subset \tilde{\Delta}_n$; moreover, $\Gamma_{n,n}(\hat{a})$ has a smooth continuation to $J$, and $\mathcal{C}^{(i)}$, $i \leq n$, deform continuously on $J$.*

The structual stability of the critical regions comes from the fact that the components of $\mathcal{C}^{(i)}$ are stacked together in a very rigid way, and their relations to the components of $\mathcal{C}^{(i-1)}$ are equally rigid. As $a$ varies over $J$, the entire structure may move up or down by amounts $>> b^{\frac{i}{2}}$, the maximum height of the components of $\mathcal{C}^{(i)}$, but it takes a relatively large horizontal displacement to slide these components past each other. A proof of Lemma 6.4 is given in Appendix B.10.

### 6.3.2 Comparing $\tau_0$-vectors for different critical curves

**Lemma 6.5** *There exists $K$ such that the following holds for all $n$: Consider $\hat{a} \in \Delta_n$ and $J = [\hat{a} - \rho^{2n}, \hat{a} + \rho^{2n}]$. Let $z^{(n)} \in \Gamma_{n,n}(\hat{a})$, $z^{(n-1)} \in \Gamma_{n-1,n-1}(\hat{a}) \cap Q^{(n-1)}(z^{(n)})$, and let $z^{(n)}(a)$ and $z^{(n-1)}(a)$ be the continuations of $z^{(n)}$ and $z^{(n-1)}$ on $J$. Then*

$$\|\frac{dz^{(n)}}{da}(a) - \frac{dz^{(n-1)}}{da}(a)\| \ < \ (Kb)^{\frac{n}{9}}.$$

From this lemma it follows inductively that $\|\frac{dz^{(n)}}{da} - \frac{dz^{(0)}}{da}\| < Kb^{\frac{1}{9}}$ where $z^{(0)}$ is a critical point of generation 0 and order 1 lying in $Q^{(0)}(z^{(n)})$. Since there is only a finite number of critical curves of generation 0 and order 1, and for them $\tau_{0,2} = 0$, Lemma 6.5 proves that the hypotheses on $\tau_0$ in Proposition 6.1 are met for curves corresponding to all $z^{(n)} \in \Gamma_{n,n}$. It remains to pass these properties to critical curves of higher order.

**Lemma 6.6** *Let $m > n$, $\hat{a} \in \Delta_m$, and let $z^m \in \Gamma_{n,m}(\hat{a})$ be the updating of $z^n \in \Gamma_{n,n}(\hat{a})$ to order $m$. Then for all $a \in [\hat{a} - \rho^{2m}, \hat{a} + \rho^{2m}]$,*

$$\|\frac{dz^m}{da}(a) - \frac{dz^n}{da}(a)\| < (Kb)^{\frac{n}{4}}.$$

Lemmas 6.5 and 6.6 are proved in Appendix B.10.

## 6.4 Dynamics of critical curves

We fix a parameter interval $J$ and a critical point $z_0$ which we assume can be smoothly continued to all of $J$. As usual, let $\gamma_i(a) = z_i(a)$. The purpose of this subsection is to make precise the parallel between the dynamics of $\gamma_0 \to \gamma_1 \to \gamma_2 \to \cdots$ and the action of $T_a^i$ on $\partial R_0$. Let $\tau_i(a) = \frac{d\gamma_i}{da}(a)$.

**Lemma 6.7** *There is a small number $k(\delta) > 0$ such that for all $i >$ some $i_0$, if $\gamma_i(a) \notin \mathcal{C}^{(0)}$ and $|\mathrm{slope}(\tau_i(a))| < k(\delta)$, then (i) $|\mathrm{slope}(\tau_{i+1}(a))| < k(\delta)$; (ii) $\tau_{i+1}(a) \approx DT_a(\gamma_i(a))\tau_i(a)$. Thus $\gamma_i$ with $|\mathrm{slope}(\tau_i)| < k(\delta)$ grows exponentially in length as long as it stays outside of $\mathcal{C}^{(0)}$.*

**Proof:** (ii) is evident once $\|\tau_i\|$ is sufficiently large. By Proposition 6.1, this happens after some $i_0$. (i) is a consequence of (ii) and Lemma 2.7. The exponential growth follows from Lemma 2.8. $\square$

We assume $(a, b)$ is sufficiently near $(a^*, 0)$ that $z_s$ remains outside of $\mathcal{C}^{(0)}$ for $> i_0$ iterates.

Next we allow $\gamma_i$ to intersect $\mathcal{C}^{(0)}$. For each fixed $a$, we have introduced in Sections 3–5 definitions of distance to the critical set, binding point, bound period, etc. To emphasize their dependence on $a$, we write $d_{\mathcal{C}(a)}(\cdot)$, $\phi_a(\cdot)$ and $p_a(\cdot)$ when referring to

definitions that belong to the map $T_a$. Even for a fixed map, these quantities depend sensitively on the location of the point in question; vertical displacements of $z$, for example, may dramatically change $\phi_a(z)$. In the "dynamics" of critical curves, the problem is all the more delicate, for not only does $z_i(a)$ move with $a$, the entire critical set moves as well. The goal of the next few lemmas is to establish some viable notions of $d_{\mathcal{C}}(\cdot)$ and bound/free states that work in a coherent fashion for all points in $\gamma_i$.

We assume for the rest of this subsection that

(i) $J \subset \tilde{\Delta}_{K\alpha n}$, so that for each $a$ the binding structure is in place for points with $d_{\mathcal{C}(a)}(\cdot) > e^{-\alpha n}$;

(ii) $z_0$ obeys (IA2) and (IA4) up to time $n$, and

(iii) all time indices are $\leq n$.

In the next lemma, we let $|\cdot - \cdot|_h$ denote the horizontal distance between two points, and assume for simplicity that $\gamma_i$ is contained in one component of $\mathcal{C}^{(0)}$.

**Lemma 6.8** *Suppose* $|\text{slope}(\tau_i)| < k(\delta)$. *Then there exists* $\bar{z} \in \mathcal{C}^{(0)}$ *such that whenever* $d_{\mathcal{C}(a)}(\gamma_i(a)) > \frac{1}{2}e^{-\alpha i}$,

$$\big||\gamma_i(a) - \bar{z}|_h - d_{\mathcal{C}(a)}(\gamma_i(a))\big| < Ke^{-ci} d_{\mathcal{C}(a)}(\gamma_i(a)).$$

Thus we may put the partition $\mathcal{P}_{\gamma_i,\bar{z}}$ on $\gamma_i$ and define $d_{\mathcal{C}}(\cdot) = |\cdot - \bar{z}|_h$ (the precise definition of $d_{\mathcal{C}}(\gamma_i(a))$ is irrelevant for $a$ with $d_{\mathcal{C}(a)}(\gamma_i(a)) < \frac{1}{2}e^{-\alpha i}$).

**Lemma 6.9** *Let* $\gamma_i$ *be as above. We assume further that* $z_i(a)$ *is a free return for every* $a$. *Then for each* $\omega_0 = I_{\mu j} \in \mathcal{P}_{\gamma_i,\bar{z}}$ *with* $|\mu| < \alpha i$, *there exists* $\tilde{p} = \tilde{p}(\omega_0) < K|\mu|$ *such that for all* $a$, $a'$ *with* $z_i(a), z_i(a') \in \omega_0$,

*(a)* $|z_{i+j}(a) - z_{i+j}(a')| < e^{-\beta j}$ *for* $j \leq \tilde{p}$;

*(b)* $z_{i+\tilde{p}}$ *is out of all fold periods,* $|\text{slope}(\tau_{i+\tilde{p}})| < k(\delta)$ *and* $|\omega_{\tilde{p}}| \geq \frac{1}{\mu^2}e^{-\beta K|\mu|}$;

*(c)* $\|w_{i+\tilde{p}}\| > K^{-1}e^{\frac{\tilde{p}}{3}}\|w_i\|$, *and* $\|\tau_{i+\tilde{p}}\| > K^{-1}e^{\frac{\tilde{p}}{3}}\|\tau_i\|$.

Lemma 6.9 allows us to define a natural notion of bound/free states for the curves $\gamma_i$ that agrees essentially with the dynamical notion previously defined for each $z_i(a)$.

**Proposition 6.2** *We assume the following hold for all* $a \in J$ *and* $i \leq n$:

*(i) for each* $i$, *the entire segment* $\gamma_i$ *is bound or free simultaneously, and* $\gamma_i$ *is contained in three contiguous* $I_{\mu j}$*'s at all free returns;*

*(ii)* $\gamma_n$ *is a free return.*

*Then there exists* $K$ *(independent of* $\gamma_0$ *or* $n$*) such that for all* $a, a' \in J$,

$$\frac{1}{K} \leq \frac{\|\tau_n(a)\|}{\|\tau_n(a')\|} \leq K.$$

Lemmas 6.8 and 6.9 are proved in Appendix B.11. Proposition 6.2 is proved in Appendix B.12.

## 6.5 Deletions on account of (IA4)

Let $J \subset \tilde{\Delta}_{3K\alpha N}$, and let $z_0$ be a critical point with a smooth continuation on $J$. We assume that for all $a \in J$, (IA2) and (IA4) hold up to time $N$, and that $\gamma_N \approx I_{\mu j}$ for some $\mu$ with $|\mu| < \alpha N$. Let

$$E_{N,z_0} := \{a \in J : \exists n,\ N < n \leq 3N, \text{ such that (IA2) is satisfied up to time } n \\ \text{and (IA4) is violated at time } n\}.$$

The set $E_{N,z_0}$ consists of parameters for which $z_i$ has an abnormally high frequency of close returns between times $N$ and $3N$.

**Proposition 6.3** *Given $\varepsilon > 0$, $\exists \delta_0 = \delta_0(\varepsilon)$ such that if $\delta < \delta_0$ and $(a, b)$ is sufficiently near $(a^*, 0)$, then*

$$|E_{N,z_0}| < e^{-\varepsilon n}|J|.$$

A 1-dimensional version of this result is proved in [BC2], page 81-86. After the groundwork in Sect. 6.4, the adaptation of this result to our setting is straightforward.

## 6.6 Estimating $|\Delta|$

The initial parameter set $\Delta_0$ is chosen as follows. Let $C = \{x_i\}$ be the critical set of $f$, and let $\delta_1$ be the minimum distance between $C$ and $f^n x_i,\ n > 0$. We assume that $\delta_1 >> \delta$. Let $n_0$ be the number of iterates the critical orbits of $T_{a,b}$ are required to stay outside of $\mathcal{C}^{(0)}$. We assume $n_0$ is as large as need be and prespecified. Then there exists $\varepsilon > 0$ such that for all $a \in [a^* - \varepsilon, a^* + \varepsilon]$ and for all small enough $b$, the first $n_0$ iterates of all the generation 0 critical points stay $> \frac{\delta_1}{2}$ away from $\mathcal{C}^{(0)}$. We let $\Delta_0 = [a^* - \varepsilon, a^* + \varepsilon]$, and let $b$ be fixed in the rest of the discussion.

We recall briefly the induction process: For $n = n_0, n_0+1, \cdots, 3N_0$ where $N_0 = [\frac{1}{\theta}]$, we consider for each fixed $a \in \Delta_0$ critical orbits of generation 0 and make deletions according to (IA2) and (IA4). By continuously updating $\Gamma_0$, we show in Sections 3–5 and 6.3.1 that orbits of $\Gamma_0$ can use their own histories for binding, defining a sequence of shrinking parameter sets $\Delta_n := \{a \in \Delta_0 : z_i(a)$ obeys (IA2) and (IA4) for all $z_0 \in \Gamma_0$ and $i \leq n\}$. At time $3N_0$, $\Gamma_{3\theta N_0}$ is introduced for $a \in \Delta_{3N_0}$, and deletions are made for orbits originating from $\Gamma_{3\theta N_0}$ up to time $3^2 N_0$.

For purposes of estimating the measure of parameters deleted in the first $3N_0$ iterates, we consider one $\hat{z}_0 \in \Gamma_0$ at a time and estimate the set of parameters discarded on account of $\hat{z}_0$ alone. The argument for $3^i N_0 < n \leq 3^{i+1} N_0,\ i \geq 1$, is identical except it has to be made for a larger set of critical points. For simplicity, let us first *decouple* the situation, i.e. pretend(!) while considering $\hat{z}_0$ that no deletions are made for any other $z_0 \in \Gamma_0$.

Let $\hat{z}_0$ be fixed in the next two paragraphs, and let $\hat{\gamma}_0$ be the curve $a \mapsto \hat{z}_0(a),\ a \in \Delta_0$. We "iterate" $\hat{\gamma}_0$ until it gets into $\mathcal{C}^{(0)}$. More precisely, let $i_0(\cdot)$ be the first time

a point enters $\mathcal{C}^{(0)}$. Then $i_0$ is a function on $\Delta_0$, and with $f_{a^*}$ being a Misiurewicz map and $\Delta_0$ chosen as in the first paragraph, it is easy to modify $i_0$ slightly so that either $\{i_0 = n\} = \emptyset$ or $\hat{\gamma}_n|\{i_0 = n\}$ contains $I_{\pm\mu^*}$, one of the outermost $I_\mu$. (This is to ensure that part of $\hat{\gamma}_n|\{i_0 = n\}$ will be retained.) We partition $\hat{\gamma}_n|\{i_0 = n\}$ into $I_{\mu j}$, letting each element of this partition play the role of the start-up segment $\omega_N$ in the model problem in Sect. 6.1. These segments are iterated independently, partitioned at free returns, and the deletion process begins. Care is taken to retain segments of full length each time something is discarded. For (IA2) we follow the procedure described in Sect. 6.1. For (IA4), it is clear from the 1-dimensional proof in [BC2] that segments of full length are retained at each stage. The technical justifications for treating $\gamma_i$ as phase curves are given in Sect. 6.4.

The procedure in the last paragraph defines for each $n$ a set $\Delta_{\hat{z}_0,n} := \{a \in \Delta_0 : a$ is retained through step $n\}$ and a partition $\mathcal{Q}_{\hat{z}_0,n}$ of $\Delta_{\hat{z}_0,n}$. Aside from $\{i_0 > n\}$, the elements of $\mathcal{Q}_{\hat{z}_0,n}$ are intervals $J$ such that $\hat{\gamma}_n|J$ in its last free return prior to time $n$ is a whole $I_{\mu j}$. From our estimates in Sect. 6.1, Proposition 6.2 (distortion estimate) and Proposition 6.3 (large deviation), it follows that there exist $\alpha_1 > 0$ and $K > 1$ such that for all $k,\ n$,

$$|(\Delta_0 \setminus \Delta_{\hat{z}_0,n}) \cap \{i_0 = k\}| \ \leq \ Ke^{-\alpha_1 k}\ |\{i_0 = k\}|.$$

In particular,

$$|\Delta_0 \setminus \Delta_{\hat{z}_0,n}| \ \leq \ Ke^{-\alpha_1 n_0}\ |\Delta_0|. \tag{11}$$

The discussion in the last two paragraphs applies to every $z_0 \in \Gamma_0$ – under the same erroneous assumption that deletions due to distinct critical orbits do not interfere with each other. We now remove this assumption:

Suppose all is well through time $n-1$. Let $J \in \mathcal{Q}_{\hat{z}_0,n-1}$ be such that all or part of $\hat{\gamma}_n|J$ makes a free return. (There is no problem if $\hat{\gamma}_n|J$ is in the middle of a free period or a bound period.) In order to continue, we need to know that the necessary binding structure exists for all $a \in J$. Observe that $J$ is not necessarily contained in $\Delta_{n-1} := \cap_{z_0\in\Gamma_0}\Delta_{z_0,n-1}$. Indeed, it may have been deleted in its entirety before time $n$ without $\hat{z}_0$ knowing about it. If that is the case, we should not have been looking at it in the first place (and hence no parameter is deleted on account of $\hat{z}_0$ at this step – or thereafter). If $J \cap \Delta_{n-1} \neq \emptyset$, we claim that $J \subset \tilde{\Delta}_{K\alpha n}$, so that the necessary binding structure for the part of $\hat{\gamma}_n|J$ to be retained is in place. The claim above is verified as follows: since $|\hat{\gamma}_n| < 1$, we have, by (IA4) and Proposition 6.1, $|J| < \lambda_1^{-1}e^{-cn}$. With $\lambda_1^{-1}e^{-cn} << \rho^{2K\alpha n}$, we are guaranteed by Lemma 6.4 that $J \subset \tilde{\Delta}_{K\alpha n}$. (Note that $J$ may not be contained in $\tilde{\Delta}_n$.) Thus the estimate (11) remains valid even as we take into consideration deletions due to other $z_0 \in \Gamma_0$.

The total measure deleted, therefore, is estimated by

$$\sum_{i=0}^{\infty} |\Delta_{3^i N_0} \setminus \Delta_{3^{i+1}N_0}| \ \leq \ \mathrm{card}(\Gamma_0)Ke^{-\alpha_1 n_0}|\Delta_0| \ + \ \sum_{i=1}^{\infty} \mathrm{card}(\Gamma_{3^i\theta N_0})Ke^{-\alpha_1 3^i N_0}|\Delta_0|.$$

To estimate card($\Gamma_{\theta N}$), let $I_1, \cdots, I_r$ be the monotone intervals of $f$, and let $K_0 = \max_i\{$ number of $I_j$ counted with multiplicity : $I_j \cap f(I_i) \neq \emptyset\}$.

**Lemma 6.10**

$$\mathrm{card}(\Gamma_{\theta N}) < K_0^{\theta N}$$

**Proof** Partition $\partial R_k$ into segments by orbits of critical points of generation $\leq k$. Then each segment has at most one free component, and each free component meets $\leq K_0$ of the monotone intervals, giving rise to $\leq K_0$ new critical points. For more details, see Sect. 9.1. $\square$

We conclude that the fraction of $\Delta_0$ deleted tends to 0 as $n_0 \to \infty$ and $b \to 0$.

**In the remainder of this paper, $T$ is assumed to be $T_{a,b}$ where $(a, b)$ is a pair of "good" parameters, i.e. $(a, b) \in \Delta$ where $\Delta$ is as in Theorem 1.**

# 7 Nonuniform Hyperbolic Behavior

Recall that $\Gamma$ is the set to which $\Gamma_{\theta N}$ converges as $N \to \infty$. One of the properties guaranteed by parameter selection is that orbits starting from $\Gamma$ have some hyperbolic behavior (Theorem 1(2)(ii)). The purpose of this section is to show that this behavior is passed on to a large set of points on the attractor and in the basin, proving Theorem 2 except for the assertion in (1)(iii), the proof of which we postpone to Sect. 10.4.

## 7.1 Control and hyperbolicity of non-critical orbits

We recapitulate the ideas developed in Sections 3–5 with a view toward proving hyperbolicity for an arbitrary (non-critical) orbit. Given arbitrary $z_0 \in R_0$, we let

$$0 \;\leq\; n_1 \;<\; n_1 + p_1 \;\leq\; n_2 \;<\; n_2 + p_2 \;\leq\; n_3 \;<\; \cdots$$

be such that $z_{n_j} \in \mathcal{C}^{(0)}$ and is bound to a suitable point in $\Gamma$, $p_j$ is the ensuing bound period, and $n_{j+1}$ is the first return after $n_j + p_j$. Then:

(1) During its free periods, i.e. between times $n_j + p_j$ and $n_{j+1}$, the orbit is outside of $\mathcal{C}^{(0)}$, where $DT^i$ is essentially uniformly hyperbolic (Lemma 2.8).

(2) During its bound periods, i.e. between times $n_j$ and $n_j + p_j$, $DT^i(z_{n_j})$ copies the derivative of its guiding orbit from $\Gamma$ (see (IA6)), which has been guaranteed through parameter selection to have some form of hyperbolicity ((IA4)).

(3) The concatenation of hyperbolic segments, however, need not result in a hyperbolic orbit, for the direction expanded at the end of one segment may be near the contractive direction of the next. Indeed, this happens at times $n_j$, when there is a "confusion" of stable and unstable directions, leading to a loss of hyperbolicity (see Sect. 3.1).

(4) The properties that guarantee that hyperbolicity is preserved through these concatenations are precisely the *h-relatedness* and *correct splitting* properties at free returns. At time $n_j$, the correct splitting of an expanded vector limits the magnitude of the loss (Lemma 2.12 and Sect. 3.3.2), while the h-relatedness of $z_{n_j}$ to some $\hat{z} \in \Gamma$ guarantees that the ensuing bound period is long enough for this loss to be compensated (see (IA5)).

In particular, if $z_0$ has a unit tangent vector $w_0$ such that $(z_0, w_0)$ is *controlled* by $\Gamma$ for all $n \geq 0$ in the sense of Definition 4.2, then

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DT^n(z_0)w_0\| \geq c' > 0 \tag{12}$$

where $e^{c'}$ is a lower bound of the growth rates of $b$-horizontal vectors outside of $\mathcal{C}^{(0)}$ and net derivative gains during bound periods. Assuming that the rate of growth outside of $\mathcal{C}^{(0)}$ is $> e^{\frac{c}{3}}$ where $c$ is as in Theorem 1, we may take $c' = \frac{c}{3}$. We remark that in general, the growth of $\|DT^n(z_0)w_0\|$ is not regular: without any assumptions on how close to $\Gamma$ the free returns are allowed to be, i.e. without a condition in the spirit of (IA2), the loss of hyperbolicity at time $n_j$ can be arbitrarily large; for example, the $\liminf$ in (12) can be negative.

Recall that to establish control of $(z_0, w_0)$, it suffices to look at free returns (Lemmas 4.2 and 4.5). We record below a condition at free returns that enables us to extend control through another bound-free cycle. Lemma 7.1 plays a crucial role in all the results in this section. First, we identify certain locations that are potentially problematic. For $k \geq 0$, let

$$Z^{(k)} := \{z \in \mathcal{C}^{(k)} : \ d_{\mathcal{C}}(z) < b^{\frac{k}{20}}\}.$$

**Lemma 7.1** *Let $z_0$ be an arbitrary point in $R_0$, $w_0$ an arbitrary unit vector, and suppose that $(z_0, w_0)$ is controlled by $\Gamma$ up to time $k-1$. Let $z_k$ be a free return. If $z_k \in \mathcal{C}^{(i)} \setminus Z^{(i)}$ for some $i < \frac{5}{4}k$, then $w_k$ splits correctly.*

**Proof:** The proof of this lemma is virtually identical to that of Proposition 5.2. Let $j = \min\{i, k\}$, so that $z_{k-j}$ makes sense. (The reason we allow $i$ to exceed $k$ has to do with the way this lemma is used.) Claims 5.1-5.3 in Proposition 5.2 continue to be valid because the only requirement on $(z_0, w_0)$ is that the pair be controlled. Note that we have already controlled $\partial R_0$ and its tangent vectors for all times. The proof here differs from that in Section 5 only at the end, where under present conditions we have

$$b^{\frac{j'}{4}} \ \leq \ b^{\frac{j}{12}} \ \leq \ b^{\frac{1}{12}\frac{4}{5}i} \ << \ b^{\frac{i}{20}} \ \leq \ d_{\mathcal{C}}(z_k).$$

$\square$

## 7.2 Typical derivative behavior in the basin

Let $m$ denote the 2-dimensional Lebesgue measure.

**Proposition 7.1** *Assuming the additional regularity condition (**) in Sect. 1.2, we have*

$$m \ \{z_0 \in R_0 : z_k \in Z^{(k)} \text{ infinitely often}\} \ = 0.$$

To prove this result, we need more refined estimates on the width of $Q^{(k)}$ than that given in Lemma 4.1.

**Lemma 7.2** *There exists $K > 0$ such that if $Q^{(k)}$ is a component of $\mathcal{C}^{(k)}$, and $d_v$ is the vertical distance between the two horizontal boundaries of $Q^{(k)}$ measured anywhere along the length of $Q^{(k)}$, then*

$$(K^{-1}b)^{k+1} \ < \ d_v \ < \ (Kb)^{\frac{99}{100}k}.$$

**Proof:** First we prove the lower bound, which relies heavily on the condition (**). Let $\omega_k$ be a vertical line segment joining two points in $\partial Q^{(k)}$. For $i < k$, let $\omega_i = T^{-k+i}\omega_k$. If $\omega_0$ connects the two components of $\partial R_0$, then $d_v > (K^{-1}b)^k \cdot K^{-1}b$ since by (**), $\|DTv\| \geq K^{-1}|\det(DT)| \geq K^{-1}K_1^{-1}b$ for every unit vector $v$. If not, we will need to rule out the possibility that $\omega_0$ may be extremely short. Let $z_0, z_0' \in \omega_0 \cap \partial R_0$, and let $\gamma_0$ be the shorter of the two segments of $\partial R_0$ between $z_0$ and $z_0'$. We consider $\gamma_i := T^i\gamma_0$, and remember that points on $\partial R_0$ together with their tangent vectors are controlled (Proposition 5.1). Since $z_k$ and $z_k'$ are both free, and they do not lie on a $C^2(b)$-curve, we conclude that a critical point is created on $\gamma_i$ for some $i < k$. Let $i$ be the first time this happens. If $|z_i - z_i'| > \delta$, then $|\omega_k| > \delta(K^{-1}b)^k$. If not, then both $z_i$ and $z_i'$ are in $\mathcal{C}^{(0)}$. Since both of their bound periods have expired by time $k$, it follows from (IA5) that $d_{\mathcal{C}}(z_i)$ and $d_{\mathcal{C}}(z_i')$ are $> e^{-K(k-i)}$. We claim that $d_{\mathcal{C}}(z_i) + d_{\mathcal{C}}(z_i')$ is approximately the horizontal distance between these two points (see Lemma 9.1 for more details). This gives $|\omega_k| > 2(e^{-K}K^{-1}b)^k$.

For the upper estimate, we pick an arbitrary $z_k \in \partial Q^{(k)}$, and borrow the argument in the proof of Claim 5.1 with $j = k$, pivoting the line $L$ at $L \cap \{x = \frac{1}{100}k\}$ (instead of $L \cap \{x = k - \frac{1}{3}j\}$) as we rotate clockwise. This gives $i_0$ with $0 \leq i_0 \leq \frac{1}{100}k$ such that $\|DT^i(z_{i_0})\| > \|DT\|^{-100i}$. Iterating forward once if necessary (and possibly losing a factor of $K^{-1}$ in the last estimate), we may assume that $z_{i_0} \notin \mathcal{C}^{(0)}$, so that it lies on an integral curve $\gamma_0$ of $e_{k-i_0}$ which joins the two components of $\partial R_0$. Note that $\gamma_0$ meets $\partial R_0$ only at its end points. Iterating forward, this curve brings in two segments of $\partial R_{k-i_0}$. They must lie on the two horizontal boundaries of $Q^{(k-i_0)}(z_k)$ because $\gamma_{k-i_0}$ passes through $z_k$ and intersects no other point of $\partial R_{k-i_0}$. This proves that $d_v$ measured at $z_k$ has length at most that of $\gamma_{k-i_0}$, which by Lemma 2.3 is $< (\|DT\|^{200}b)^{k-i_0} < (Kb)^{\frac{99}{100}k}$. □

**Proof of Proposition 7.1**: By the Borel-Cantelli Lemma, it suffices to show that $\sum_k m(T^{-k}Z^{(k)}) < \infty$. We estimate $m(T^{-k}Z^{(k)})$ by

$$\begin{aligned} m(T^{-k}Z^{(k)}) &= \sum m(T^{-k}(Q^{(k)} \cap Z^{(k)})) \\ &\leq \max \frac{m(T^{-k}(Q^{(k)} \cap Z^{(k)}))}{m(T^{-k}Q^{(k)})} \sum m(T^{-k}Q^{(k)}) \end{aligned}$$

where the summations and maximum are taken over all components $Q^{(k)}$ of $\mathcal{C}^{(k)}$. Note also that $\sum m(T^{-k}Q^{(k)}) < 1$. Using Lemma 7.2 and the regularity of $\det(DT)$ in (**), we obtain

$$\begin{aligned} \frac{m(T^{-k}(Q^{(k)} \cap Z^{(k)}))}{m(T^{-k}Q^{(k)})} &\leq K^{2k} \cdot \frac{m(Q^{(k)} \cap Z^{(k)})}{m(Q^{(k)})} \\ &\leq K^{2k} \cdot \frac{(Kb)^{\frac{99}{100}k} \cdot b^{\frac{1}{20}k}}{(\frac{b}{K})^{k+1} \cdot \rho^k} \leq K^{4k}\frac{1}{b} \cdot \frac{b^{\frac{1}{25}k}}{\rho^k} \end{aligned}$$

which decreases geometrically in $k$ as desired. □

**Proof of Theorem 2(2):** Let $\xi_0 \in R_0$. From the discussion in Sect. 7.1, it follows that
$$\limsup_{n\to\infty} \frac{1}{n} \log \|DT^n(\xi_0)\| \geq \frac{c}{3}$$
holds if we are able to produce $k_0 > 0$ and a vector $w_0$ such that if $z_0 = \xi_{k_0}$, then $(z_0, w_0)$ is controlled by $\Gamma$ for all $n \geq 0$. In light of Proposition 7.1, it suffices to consider the following two cases.

*Case 1.* $\xi_k \notin Z^{(k)}$ for all $k \geq 0$. We take $k_0 = 0$ and let $w_0 = \binom{1}{0}$ if $\xi_0 \notin \mathcal{C}^{(0)}$, $w_0 = \binom{0}{1}$ if $\xi_0 \in \mathcal{C}^{(0)}$. We assume $(z_0, w_0)$ is controlled up to time $k-1$, and let $z_k$ be a free return. The hypothesis of Lemma 7.1 is verified at time $k$ as follows: Let $j$ be the largest integer such that $z_k \in \mathcal{C}^{(j)}$. Then if $j \geq k$, $i = k$ meets the requirements of Lemma 7.1 since $\xi_k \notin Z^{(k)}$; and if $j < k$, then $z_k$ must be in $\hat{Q}^{(j+1)} \setminus Q^{(j+1)}$ for some $Q^{(j+1)}$ since it is in $R_k$, and so we may take $i = j + 1$.

*Case 2.* $\xi_{k_0} \in Z^{(k_0)}$ for some $k_0$ and $\xi_k \notin Z^{(k)}$ for all $k > k_0$. Here we let $z_0 = \xi_{k_0}$ and $w_0 = \binom{0}{1}$. There is a critical point $\hat{z}$ in $Q^{(k_0)}(z_0)$ to which $z_0$ is bound for $k_1$ iterates. Since $\|DT\|^{k_1} b^{\frac{k_0}{20}} > e^{-\beta k_1}$, we have $k_1 \sim k_0\theta^{-1} >> k_0$. During this period, we may regard $(z_0, w_0)$ as controlled by $\Gamma$. For $k \geq k_1$, the situation is identical to that in Case 1 except that $z_k \in R_{k+k_0}$ and we can only guarantee $z_k \notin Z^{(k+k_0)}$. To verify the hypothesis of Lemma 7.1 for $z_k$, we proceed as above, distinguishing between the cases $j \geq k + k_0$ and $j < k + k_0$ and noting that for $k \geq k_1$, $k + k_0 < (1 + K\theta)k$. $\square$

**Remark.** The results in this paper that use (**) in Sect. 1.2 remain valid if (**) is replaced by

$(**)'$ There exist $\eta \geq 1$ and $K_1, K_2 > 0$ such that for all $z \in R_0$,
$$K_1^{-1} b^\eta \leq |\det(DT)| \leq K_2 b^\eta.$$

To Prove this, it suffices to check that Proposition 7.1 is valid under $(**)'$. Observe that the results in Sect. 2.1 are abstract, so that if $\|DT^i(z_0)\| \geq \kappa^i$ for all $i \leq n$, then $\|DT^i e_n\| \leq (Kb^\eta \kappa^{-2})^i$ for all $i \leq n$. Using this and $\|DTv\| \geq K^{-1}b^\eta$ for all $\|v\| = 1$, one checks easily that under $(**)'$, the conclusion of Lemma 7.2 is valid if $b$ is replaced by $b^\eta$. Moreover, the number $\frac{99}{100}$ can be replaced by $1 - \varepsilon_0$ for any prespecified $\varepsilon_0 > 0$. Choosing $\varepsilon_0$ such that $\varepsilon_0 \eta < \frac{1}{20}$, we check that the proof of Proposition 7.1 goes through as is.

## 7.3 Uniform hyperbolicity away from $\mathcal{C}$

Recall that
$$\Omega_\varepsilon = \{z_0 \in \Omega : \ \ d_{\mathcal{C}}(z_n) \geq \varepsilon \ \text{ for all } n \in \mathbb{Z}\}.$$
The purpose of this subsection is to prove that $\Omega_\varepsilon$ is a uniformly hyperbolic invariant set [7] for every $\varepsilon > 0$. This result together with the fact that the strength of hyper-

[7] Technically, $z^i \to z$ does not imply $d_{\mathcal{C}}(z^i) \to d_{\mathcal{C}}(z)$ when $z^i \notin \mathcal{C}^{(k)}$ and $z \in \mathcal{C}^{(k)}$, but let us assume $\Omega_\varepsilon$ is closed by taking its closure if necessary.

bolicity deteriorates as $\varepsilon \to 0$ justifies our identification of $\mathcal{C}$ as the critical set and confirms that $d_{\mathcal{C}}(\cdot)$ is a valid notion of "distance" to the critical set. The approximation of $\Omega$ by $\Omega_\varepsilon$ is a concrete example of the use of uniformly hyperbolic invariant sets to approximate systems that have (weak) hyperbolic properties. See [K] and [P] for results in the same spirit.

Proofs of uniform hyperbolicity often rely on *a priori* knowledge of invariant cones. In our setting, these cones are easily identified for $\Omega_\varepsilon$ with $\varepsilon > \sqrt{b}$; see Sect. 2.5. As $\varepsilon \to 0$, the situation becomes considerably more delicate: the stable and unstable directions at points in $\Omega_\varepsilon$ become increasingly confused, both ranging over nearly all possible directions within very small neighborhoods. Our line of proof, which does not rely on *a priori* knowledge of cones, can be formulated as follows:

Let $g : X \to X$ be a self-map of a compact metric space, and let $M : X \to GL(2, \mathbb{R})$ be a continuous map. For $x \in X$ and $n \geq 0$, we define $M^{(n)}(x) = M(g^{n-1}x) \cdots M(gx)M(x)$ and $M^{(-n)}(x) = M(g^{-n}x)^{-1} \cdots M(g^{-1}x)^{-1}$. It is clear what it means for the *cocycle* $(g, M^{(n)})$ to be *uniformly hyperbolic* (think of $g$ as a diffeomorphism and $M(x) = Dg(x)$). Since the condition of interest to us is projective in nature, we will state our result assuming that $M$ takes its values in $SL(2, \mathbb{R})$.

**Lemma 7.3** *Let $(g, M^{(n)})$ be as above. If there exist $\lambda > 1$ and $N \in \mathbb{Z}^+$ such that at each $x \in X$, there exists a unit vector $v = v(x)$ such that*

$$\|M^{(n)}(x)v\| \leq \lambda^{-n} \quad \text{for all} \quad n \geq N,$$

*then $(g, M^{(n)})$ is uniformly hyperbolic.*

**Proof:** Let $E^s(x)$ be the subspace spanned by $v(x)$, and observe that $M(x)E^s(x) = E^s(gx)$: if not, then there are two linearly independent vectors, $v_1 \in M(x)E^s(x)$ and $v_2 = v(gx)$ such that both $\|M^{(n)}(gx)v_1\|$ and $\|M^{(n)}(gx)v_2\|$ decrease exponentially as $n \to \infty$, contradicting $M \in SL(2, \mathbb{R})$. The continuity of $x \mapsto E^s(x)$ is proved similarly.

Using the uniform contraction of $M^{(N)}$ on vectors in $E^s$ and the fact that $|\det(M)| = 1$, we choose $\delta_0 > 0$ such that for all $x \in X$ and $w \neq 0 \in \mathbb{R}^2$, if $\angle(w, v(x)) < \delta_0$, then $\angle(M^{(N)}(x)w, v(g^N x)) > \frac{1}{2}\lambda^{2N}\angle(w, v(x))$. Let $C^s(x) = \{w : \angle(w, v(x)) < \delta_0\}$ and $C^u(x) = \mathbb{R}^2 \setminus C^s(x)$. We claim that $E^u(x) := \cap_{n=1}^\infty M^{(nN)}(g^{-nN}x)C^u(g^{-nN}x)$ is a 1-dimensional subspace. This is true because from the angle separation between vectors in $E^s$ and $E^u$, it follows that for all $w \in E^u$, $\|M^{(-nN)}w\|$ decreases exponentially. The $M$-invariance of $E^u$ is checked easily. $\square$

**Proposition 7.2** *For every $\varepsilon > 0$, $\Omega_\varepsilon$ is uniformly hyperbolic with*

$$\|DT^i u\| \;\geq\; K_\varepsilon^{-1} e^{c' i}$$

*for all $u \in E^u$. Here $K_\varepsilon$ is a constant depending on $\varepsilon$, and $c'$ can be taken to be $\approx \frac{c}{3}$.*

**Proof:** We fix $\varepsilon$ and let $k_\varepsilon$ be the smallest integer $k$ such that $\varepsilon > b^{\frac{k}{20}}$.

**Claim 7.1** *For every $\xi_0 \in \Omega_\varepsilon$, there exists $k(\xi_0) \leq 2k_\varepsilon$ and a unit vector $w_0$ such that if $z_0 = \xi_{k(\xi_0)}$, then for all $i > 0$,*

$$\|DT^i(z_0)w_0\| \;\geq\; e^{\frac{c}{3}i} b^{\frac{k_\varepsilon}{20}} K^{-\frac{k_\varepsilon}{10}}.$$

*Proof of Claim 7.1:* We consider separately the following cases:

*Case 1.* $\xi_i \notin \mathcal{C}^{(0)}$ for all $i \leq k_\varepsilon$. In this case we let $k(\xi_0) = 0$ and $w_0 = \binom{1}{0}$.

*Case 2.* $\xi_{i_0} \in \mathcal{C}^{(0)}$ for some $i_0 \leq k_\varepsilon$ and $\xi_{i_0+k} \notin Z^{(k)}$ for all $k \geq 0$. We let $k(\xi_0) = i_0$ and $w_0 = \binom{0}{1}$.

*Case 3.* $\xi_{i_0} \in \mathcal{C}^{(0)}$ for some $i_0 \leq k_\varepsilon$ and $\xi_{i_0+k} \in Z^{(k)}$ for some $k \geq 0$. We let $k$ be the last time this happens, and choose $k(\xi_0) = i_0 + k$, $w_0 = \binom{0}{1}$. Note that $k(\xi_0) \leq 2k_\varepsilon$.

In each of the three cases, we first show that $(z_0, w_0)$ is controlled by $\Gamma$ for all $n \geq 0$. This is done by verifying inductively at free returns the hypothesis of Lemma 7.1. The arguments are essentially the same as those for Theorem 2(2).

From the control of $(z_0, w_0)$, it follows that at free returns, $\|w_i\| > e^{\frac{c}{3}i}$. Next we consider the drop in $\|w_i^*\|$ one step later. This is given by $d_{\mathcal{C}}(z_i)$, which by the definition of $\Omega_\varepsilon$ is $\geq b^{\frac{k_\varepsilon}{20}}$. Further drops at bound returns are exponentially small. For comparisons between $w_i^*$- and $w_i$- vectors, since the fold period $\ell$ initiated at time $i$ is $\leq \frac{k_\varepsilon}{10}$, we have, for $j < \ell$, $\|w_{i+j}\| \geq K^{-\frac{k_\varepsilon}{10}} \|w_{i+\ell}\| = K^{-\frac{k_\varepsilon}{10}} \|w_{i+\ell}^*\|$. $\diamondsuit$

Let $z_0$ be as above. From Claim 7.1, the fields of most contracted directions of sufficiently high orders are defined at $z_0$, and their uniform contractive estimates are passed on to $e_\infty := \lim_n e_n$ (see Corollary 2.1). Let $v(z_0) = e_\infty(z_0)$. For other $\xi_0 \in \Omega_\varepsilon$, let $v(\xi_0) = DT^{-k(\xi_0)}(\xi_{k(\xi_0)}) v(\xi_{k(\xi_0)})$. Using the fact that $k(\xi_0) < 2k_\varepsilon$ and letting $M(z) = \frac{1}{|\det DT(z)|^{1/2}} DT(z)$, we see that the conditions of Lemma 7.3 are satisfied. Uniform hyperbolicity follows.

It remains to prove that a lower bound for $\|DT^i|E^u\|$ is as claimed. In the argument above we have produced for each $\xi_0 \in \Omega_\varepsilon$ a vector $u_0$ uniformly bounded away from $E^s(\xi_0)$ such that $\|u_i\| \geq K_\varepsilon^{-1} e^{\frac{c}{3}i}$. Since $\angle(u_n, E^u(\xi_n)) \to 0$ uniformly, we have $\|u_n\| \sim \|DT^n|E^u(\xi_0)\|$. The assertion in Theorem 2(i) on periodic points is proved similarly. $\square$

**Proof of Theorem 2(1)(ii):** We now prove that the deterioration of hyperbolicity on $\Omega_\varepsilon$ as $\varepsilon \to 0$ is not only a possibility but a fact. To do this, it suffices to produce a point $z \in \Omega_\varepsilon$ with the property that $\angle(E^u(z), E^s(z)) < K\varepsilon$. We can choose this point to be on the unstable manifold $W^u(\hat{z})$ of any $\hat{z} \in \Omega_\delta$. For $\xi_0 \in W^u(\hat{z})$, let $\tau_0$ be its unit tangent vector to $W^u(\hat{z})$,

**Claim 7.2** *For all $\xi_0 \in W^u(\hat{z})$, $(\xi_0, \tau_0)$ is controlled by $\Gamma$ for all $n \geq 0$.*

*Proof of Claim 7.2:* It suffices to prove the result for $\xi_0 \in W^u_{loc}(\hat{z})$. Suppose that $(\xi_0, \tau_0)$ is controlled up to time $k-1$, $\xi_k$ is a free return, and $\xi_k \in \mathcal{C}^{(j-1)} \setminus \mathcal{C}^{(j)}$ for some $j$. Since $\xi_{k-j} \in \Omega$, it follows that $\xi_k \in R_j$, so that $\xi_k \in \hat{Q}^{(j)} \setminus Q^{(j)}$ for some $Q^{(j)}$. If $j \le k$, then Lemma 7.1 applies directly. If not, we let $z_0 = \xi_{k-j}$ and apply Lemma 7.1 to the orbit of $(z_0, \tau_0(z_0))$. $\diamondsuit$

Let $\gamma = W^u_{\delta/2}(\hat{z})$. We will show that there exists $z \in (T^n\gamma \cap \Omega_\varepsilon)$ for some $n > 0$ such that $d_{\mathcal{C}}(z) < 2\varepsilon$. As $\gamma$ is iterated, it gets long and eventually meets the region $\{d_{\mathcal{C}}(\cdot) < \varepsilon\}$. Let $n_0$ be the first time this happens, and let $\omega_0 \subset T^{n_0}\gamma$ correspond to some $I_{\mu j}$ in the region $\{\varepsilon \le d_{\mathcal{C}}(\cdot) \le 2\varepsilon\}$. (See the beginning of Sect. 6.1 for notation.) Note that $\omega_0$ is free. We set binding for $\omega_0$ and iterate until it becomes free again at time $n_1$. We then subdivide the image into segments corresponding to $I_{\mu j}$ (by which we include pieces outside of $\mathcal{C}^{(0)}$), and let $\omega_1$ be the longest of the divided subsegments. We iterate $\omega_1$ until it becomes free again at time $n_2$. Then divide and choose $\omega_2$ to be the longest of the subsegments etc. Let $z \in \cap_{i \ge 0} T^{-(n_i - n_0)}\omega_i$. Using Corollary 4.3, we verify that $\omega_i \cap \{d_{\mathcal{C}}(\cdot) < \varepsilon\} = \emptyset$ for all $i \ge 0$, so that $z \in \Omega_\varepsilon$.

It remains to estimate $\angle(E^u(z), E^s(z))$. First, since $\tau(z)$ splits correctly, we have $\angle(E^u(z), \tau(\phi(z))) < \varepsilon_0 d_{\mathcal{C}}(z) < 2\varepsilon_0\varepsilon$. Note that $\tau(\phi(z)) = e_\infty(\phi(z))$ and $E^s(z) = e_\infty(z)$. We leave it as an easy exercise to show that $\|DT^n(z)\tau_0(z_0)\| \ge 1$ for all $n > 0$ (use Claim 7.2 and Corollary 4.3), so that at both $z$ and $\phi(z)$, $\angle(e_n, e_\infty) = \mathcal{O}(b^n)$. Let $n$ be such that $\lambda^n \sim \varepsilon$ where $\lambda$ is as in Lemma 2.2. Then $\angle(e_n(z), e_n(\phi(z))) < K\varepsilon$, and $\mathcal{O}(b^n) << \varepsilon$, proving $\angle(\tau(\phi(z)), E^s(z)) < K'\varepsilon$. This completes the proof. $\square$

# 8 Statistical Properties of SRB Measures

We follow [Y3] and [Y4], which put forward a scheme for obtaining statistical information for general dynamical systems with some hyperbolic properties. In this approach, one constructs reference sets and studies regular returns to these sets. Sufficient conditions in terms of return times are then given for various statistical properties.

In Sect. 8.1, we indicate how this setup is arranged for the class of attractors in question. For technical details on this construction, we refer the reader to [BY2], where a similar construction is carried out for the Hénon maps. SRB measures and their statistical properties are discussed in Sects. 8.2 and 8.4. A feature of the present setting is that depending on the transitivity properties of $T$, our attractor may admit multiple SRB measures.

Obviously, the method of [Y3] and [Y4] gives information only on orbits that pass through the reference sets constructed. To complete the picture, we prove in Sect. 8.3 that all SRB measures are captured by our reference sets, and Lebesgue-almost every initial condition in the basin is accounted for.

## 8.1 Positive-measure horseshoes with infinitely many branches and variable return times

In [Y3], a unified way of looking at nonuniformly hyperbolic systems is proposed. This dynamical picture requires that one constructs a reference set and a return map with Markov properties. The purpose of this subsection is to recall this construction in the context of the maps under consideration, and to give a summary of the facts needed in the discussion to follow.

### 8.1.1 Construction of reference set

Let $\{x_1, \cdots, x_r\}$ be the set of critical points of $f$. Our reference set $\Lambda$ is the disjoint union of $2r$ Cantor sets $\Lambda_1^\pm, \cdots, \Lambda_r^\pm$ where $\Lambda_i^+$ and $\Lambda_i^-$ are located in the component of $\mathcal{C}^{(0)}$ containing $(x_i, 0)$, one on each side of $(x_i, 0)$. We define $\Lambda_i^+$ (respectively $\Lambda_i^-$) by specifying two transversal families of curves $\Gamma_i^{+,s}$ and $\Gamma_i^{+,u}$ and letting

$$\Lambda_i^+ = \{z \in \gamma^u \cap \gamma^s : \gamma^u \in \Gamma_i^{+,u},\ \gamma^s \in \Gamma_i^{+,s}\}.$$

The family $\Gamma_i^{+,s}$ (no relation to the critical set $\Gamma_i$ in Sections 3–6) is defined as follows. Let $\mathcal{P}$ be the partition in Sect. 6.1 centered at $(x_i, b) \in \partial R_0$. (To simplify notation, $\partial R_0$ in this section refers to the top boundary of $R_0$.) Let $\omega_0 \subset \partial R_0$ be the outermost $I_{\mu j}$ on the right, and let $\omega_\infty = \{z_0 \in \omega_0 : d_{\mathcal{C}}(z_n) > \delta e^{-\alpha n}$ for all $n \geq 0\}$. Letting $m_\gamma(\cdot)$ denote the measure on a curve $\gamma$ induced by arc length, it is proved in Sect. 6.1 that $m_{\omega_0}(\omega_\infty) > 0$. For every $z_0 \in \omega_\infty$, since $\|DT^i(z_0)\tau_0\| \geq \delta e^{\frac{cn}{3}}$ for all $n \geq 0$ (use (IA5) and the definition of $\omega_\infty$), there is a stable curve of every order passing through it. These curves converge to a stable curve $\gamma^s(z_0)$ of infinite order (Sect. 2.1). Moreover, $\gamma^s(z_0)$ has slope $> K^{-1}\delta$ and connects the two boundaries of $R_0$. We define $\Gamma_i^{+,s} := \{\gamma^s(z_0) : z_0 \in \omega_\infty\}$.

To define $\Gamma_i^{+,u}$, we first let $\tilde{\Gamma}_i^{+,u}$ be the set of all free segments $\gamma$ of $\partial R_n$, all $n \geq 0$, such that $\gamma$ is three times as long as $\omega_0$ and has its midpoint vertically aligned with that of $\omega_0$. Let $\Gamma_i^{+,u}$ be the set of curves that are pointwise limits of sequences in $\tilde{\Gamma}_i^{+,u}$. We remark that since the curves in $\tilde{\Gamma}_i^{+,u}$ are $C^2(b)$, their slopes as functions in $x$ form an equicontinuous family. This implies that the curves in $\Gamma_i^{+,u}$ are at least $C^{1+Lip}$, and that the tangent vectors of curves in $\tilde{\Gamma}_i^{+,u}$ converge uniformly to the tangent vectors of curves in $\Gamma_i^{+,u}$.

Recalling that $\Lambda_i^+$ and $\Lambda_i^-$ are the Cantor sets that straddle $x_i$, we may, for convenience, choose $\Gamma_i^{-,s}$ and $\Gamma_i^{+,s}$ in such a way that their elements are paired, i.e. the $T$-image of each element in $\Gamma_i^{-,s}$ lies on a stable curve containing the $T$-image of an element of $\Gamma_i^{+,s}$, and *vice versa*.

This completes the construction of $\Lambda = \cup_{i=1}^r \Lambda_i^\pm$. A similar construction is carried out for the Hénon maps in [BY2], Sects. 3.1-3.4.

### 8.1.2 Structure of return map

Next we define a return map $T^R : \Lambda \to \Lambda$ with the following properties: Topologically, $T^R : \Lambda \to \Lambda$ has the structure of an infinite horseshoe. For simplicity of notation, we write $\Lambda_i = \Lambda_i^+$ or $\Lambda_i^-$. A set $X \subset \Lambda_i$ is called an *s-subset* of $\Lambda_i$ if there exists a subcollection of $\Gamma \subset \Gamma_i^s$ such that $X = \{z \in \gamma^s \cap \gamma^u : \gamma^s \in \Gamma,\ \gamma^u \in \Gamma_i^u\}$; *u-subsets* are defined similarly. If $X$ is an $s$-subset of $\Lambda_i$, we say $X = \Lambda_i \ mod\ 0$ if $m_{\partial R_0}(\Lambda_i \triangle X) = 0$.

**Lemma 8.1** *There is a map $T^R : \Lambda \to \Lambda$ with the following properties: every $\Lambda_i$ has a collection of pairwise disjoint s-subsets $\{\Lambda_{i,j}\}_{j=1,2,\cdots}$ with $\Lambda_i = \cup_j \Lambda_{i,j}$ mod 0 such that for each $j$,*
- *$T^R|\Lambda_{i,j} = T^{n_{i,j}}|\Lambda_{i,j}$ for some $n_{i,j} \in \mathbb{Z}^+$;*
- *$T^R(\Lambda_{i,j})$ is a u-subset of $\Lambda_k$ for some $k = k(i,j)$.*

We stress that the partition of $\Lambda_i$ into $\{\Lambda_{i,j}\}$ is an infinite one, and that the return times $n_{i,j}$ are not bounded. The **return time function** $R : \Lambda \to \mathbb{Z}^+$ is defined to be $R|\Lambda_{i,j} = n_{i,j}$. As we will see, the tail of this function, that is, the distribution of its large values, plays a crucial role in determining the statistical properties of the system. Note that $T^R$ is not necessarily the first return map; we have settled for possibly larger return times in favor of a Markov structure. Lemma 8.1 corresponds to Proposition A(1) in [BY2]; its proof is given in Sects. 3.4 and 3.5 of [BY2].

### 8.1.3 Two important analytic estimates

Technical estimates corresponding to (P1)-(P5) in [Y3] or Proposition B of [BY2] are needed. Referring the reader to Section 5 of [BY2] for their precise statements and proofs, we state below two of the most relevant facts.

**Lemma 8.2 (Distortion estimate for controlled segments)** *There exists $K > 0$ such that the following holds: Let $\gamma_0$ be a curve and $\tau_0$ its unit tangent vectors. We assume that*

*(i) for all $z_0 \in \gamma_0$, $(z_0, \tau_0)$ is controlled up to time $n-1$;*

*(ii) $\gamma_i$ is bound or free simultaneously for each $i$, and $\gamma_i$ is contained in three contiguous $I_{\mu j}$ at all free returns;*

*(iii) $\gamma_n$ is a free return.*

*Then for all $z_0, z_0' \in \gamma_0$,*

$$\frac{1}{K} \le \frac{\|\tau_n(z_0)\|}{\|\tau_n(z_0')\|} \le K.$$

The proof is similar to that of Proposition 6.2 (it is, in fact, a little simpler) and will be omitted. In the construction of $T^R : \Lambda \to \Lambda$, it is important to arrange that $\gamma_0$, the shortest subsegment of $\partial R_0$ that spans $\Lambda_{i,j}$ in the $u$-direction, satisfies (ii) above up to time $n_{i,j}$.

Let $\cup\Gamma_i^s := \cup\{z \in \gamma^s : \gamma^s \in \Gamma_i^s\}$. If $\gamma$ and $\gamma'$ are curves transversal to the elements of $\Gamma_i^s$ and intersecting them, we define $\psi : \gamma \cap (\cup\Gamma_i^s) \to \gamma'$ by sliding along the curves in $\Gamma_i^s$, and say $\Gamma_i^s$ is *absolutely continuous* if for every pair of $C^2(b)$-curves $\gamma$ and $\gamma'$ as above, $\psi$ carries sets of $m_\gamma$-measure zero to sets of $m_{\gamma'}$-measure zero. Recall that if $\gamma$ is the subsegment of $\partial R_0$ in $\Gamma_i^u$, then $m_\gamma(\gamma \cap (\cup\Gamma_i^s)) > 0$; in particular, the definition above is not vacuous.

**Lemma 8.3 (Absolute continuity of $\Gamma_i^s$)** *$\Gamma_i^s$ is absolutely continuous with*

$$\frac{1}{K} \;<\; \frac{d}{dm_{\gamma'}}\psi_*(m_\gamma | \cup\Gamma_i^s) \;<\; K \qquad \text{on } \gamma' \cap (\cup\Gamma_i^s).$$

Except for one minor technical difference, the proof of Lemma 8.3 is identical to that of Sublemma 10 in Section 5 of [BY2]: in the latter, the transversals are taken to be curves in $\tilde{\Gamma}_i^u$, whereas we need them to be arbitrary $C^2(b)$ curves here. Clearly, it suffices to show that Sublemma 10 of [BY2] is valid with $\gamma \in \tilde{\Gamma}_i^u$ and $\gamma'$ arbitrary $C^2(b)$, and for that we need distortion estimates for the $\tau_i$-vectors on certain subsegments of $\gamma'$ ($\omega'$ in the proof of Sublemma 10). We have them because these subsegments are connected to subsegments of $\gamma$ by (temporary) stable curves, and the corresponding $\tau_i$-vectors are comparable.

#### 8.1.4 Tail of return times

Finally we state an estimate on which the statistical properties of $T$ depend crucially. Its proof is identical to that of Proposition A(4) in [BY2].

**Lemma 8.4** *There exists $K$ and $\theta_0 < 1$ such that for every $\Lambda_i$,*

$$m_{\partial R_0}\{z \in \partial R_0 \cap \Lambda_i : R(z) > n\} < K\theta_0^n.$$

## 8.2 SRB measures

#### 8.2.1 Construction of SRB measures

We describe below a recipe for constructing SRB measures using the reference sets $\{\Lambda_i^\pm\}$. For the definition of SRB measures, see Sect. 1.3. For more details on the technical justification of the steps below, see [Y3] or [BY2], Sect. 6.2. The construction consists of three steps.

*Step 1. Construction of a $T^R$-invariant measure $\nu$ on $\cup\Lambda_k^\pm$ with absolutely continuous conditional measures on the leaves of $\Gamma^u := \cup_k\Gamma_k^{\pm,u}$.* We fix some $\Lambda_i = \Lambda_i^+$ or $\Lambda_i^-$, and let $m_0 = m_{\partial R_0} \mid (\Lambda_i \cap \partial R_0)$. Let $\nu$ be an accumulation point of the sequence of measures

$$\frac{1}{n}\sum_{j=0}^{n-1}(T^R)_*^j m_0, \quad n = 1, 2, \cdots.$$

Then $\nu$ is a $T^R$-invariant measure. By Lemma 8.2, the conditional measures of $(T^R)^j_* m_0$ on the curves of $\tilde{\Gamma}^u := \cup_k \tilde{\Gamma}^{\pm,u}_k$ have uniformly bounded densities. From Lemma 8.2 and the Markov property of $T^R$ (see Sect. 8.1.2), it follows that for $\gamma \in \tilde{\Gamma}^u$, the conditional densities of $(T^R)^j_* m_0$ on $\gamma$ when restricted to $\gamma \cap (\cup \Gamma^s)$ are uniformly bounded away from 0. These properties are passed on to the conditional measures of $\nu$ on the leaves of $\Gamma^u$. (The curves in $\Gamma^u$ are pairwise disjoint except possibly for a countable number of pairs; this is nothing more than a technical nuisance.)

*Step 2. Construction of a $T$-invariant probability measure $\mu$ given $\nu$.* It follows from the bounded densities of $\nu$, Lemma 8.3 and Lemma 8.4 that $\int_\Lambda R d\nu_0 < \infty$. Let

$$\mu \;=\; \frac{1}{\int R d\nu_0} \sum_{j=0}^{\infty} T^j_*(\nu_0 \mid \{R > j\}).$$

It is straightforward to check that $\mu$ is a $T$-invariant probability measure.

*Step 3. Proof of SRB property.* Let $\mu$ be as in Step 2. First we check that $T$ has a positive Lyapunov exponent $\mu$-a.e. At $z_0 \in (\cup \Gamma^s) \cap (\cup \tilde{\Gamma}^u)$, let $\tilde{\tau}(z_0)$ be a unit tangent vector to $\tilde{\Gamma}^u(z_0)$, the $\tilde{\Gamma}^u$-curve through $z_0$. Just as on $\omega_\infty$, we have $\|DT^n(z_0)\tilde{\tau}\| \geq \delta e^{\frac{cn}{3}}$ for all $n \geq 0$. This uniform growth is passed on to the tangent vectors $\tau$ to $\Gamma^u$-curves at every $z \in \Lambda = \cup_k \Lambda^\pm_k$. The existence of a positive Lyapunov exponent $\mu$-a.e. follows from the fact that the orbit of $\mu$-almost every point passes through $\Lambda$. General nonuniform hyperbolic theory (see e.g. [P] or [R4]) then tells us that stable and unstable manifolds exist $\mu$-a.e.

To prove that $\mu$ is an SRB measure, we need to show that its conditional measures on unstable manifolds are absolutely continuous. Since $\mu$ is the sum of forward images of $\nu$, it suffices to prove this for $\nu$. We know from Step 1 that $\nu$ has absolutely continuous conditional measures on the leaves of $\Gamma^u$. Thus it remains to prove

**Claim 8.1** *For $\nu$-a.e. $z_0$, $\Gamma^u(z_0)$ is a local unstable manifold, i.e.*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\xi_0 \in \Gamma^u(z_0)} |\xi_{-n} - z_{-n}| \;<\; 0.$$

*Proof of Claim 8.1:* From the construction of $\nu$, it follows that that for $\nu$-a.e. $z_0 \in \Lambda$, there is a sequence of $\tilde{\Gamma}^u$-curves $\{\tilde{\gamma}_i\}$ such that $\tilde{\gamma}_i \to \Gamma^u(z_0)$. Let $n_i$ be such that $T^{-n_i}\tilde{\gamma}_i \subset \partial R_0$. Since $\tilde{\gamma}_i$ is free, we have that for all tangent vectors $\tilde{\tau}$ of $\tilde{\gamma}$, $\|DT^{-n}\tilde{\tau}\| < e^{-c''n}$ for some $c'' > 0$ and $0 < n \leq n_i$ (Proposition 5.1 and Lemma 4.8). These uniform estimates for backward iterates of $T$ are passed on to all tangent vectors of $\Gamma^u(z_0)$, proving that it is a local unstable manifold of $z_0$. $\diamondsuit$

### 8.2.2 Ergodic decomposition of SRB measures

We begin by considering the ergodic decompositions of the $T^R$-invariant measures constructed in Step 1 in Sect. 8.2.1.

**Definition 8.1** *Let $g : X \to X$ be a continuous map of a compact metric space, and let $\nu$ be a $g$-invariant Borel probability measure on $X$. We say $z \in X$ is* **generic** *or* **future-generic** *with respect to $\nu$ if for every continuous function $\varphi : X \to \mathbb{R}$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(z_i) \to \int \varphi d\nu.$$

Let $\mathcal{M}(T^R)$ be the set of all normalized invariant measures constructed in Step 1 of Sect. 8.2.1. Let $\nu \in \mathcal{M}(T^R)$, and suppose that $\nu(\Lambda_i) > 0$ for some $\Lambda_i = \Lambda_i^+$ or $\Lambda_i^-$. From the positivity of the conditional densities of $\nu$ on $\Lambda_i \cap \gamma$, Lemma 8.3, and a standard argument due to Hopf, we know that there is an ergodic component $\nu^e$ of $\nu$ such that

(i) $\nu$-a.e. $z \in \Lambda_i$ is generic with respect to $\nu^e$;

(ii) for every $C^2(b)$-curve $\gamma$, $m_\gamma$-a.e. $z \in \gamma \cap (\cup \Gamma_i^s)$ is generic with respect to $\nu^e$.

We abbreviate this by saying $\nu^e$ "occupies" $\Lambda_i$.

Let $\mathcal{M}_e(T^R)$ denote the set of normalized ergodic components of measures in $\mathcal{M}(T^R)$. Then each $\Lambda_i^+$ (resp. $\Lambda_i^-$) is occupied by an elememt of $\mathcal{M}_e(T^R)$. Because the stable curves of $\Lambda_i^+$ and $\Lambda_i^-$ are joined, $\Lambda_i^-$ and $\Lambda_i^+$ are in fact occupied by the same element of $\mathcal{M}_e(T^R)$. Thus the cardinality of $\mathcal{M}_e(T^R)$ is $\leq r$.

To further study the structure of $\mathcal{M}_e(T^R)$ we borrow some ideas from finite state Markov chains. Let $\Lambda_i^\pm := \Lambda_i^+ \cup \Lambda_i^-$. We think of each the sets $\Lambda_i^\pm, i = 1, \cdots, r$, as a state, and write "$i \to j$" if $T^R(\Lambda_i^\pm) \cap \Lambda_j^\pm \neq \emptyset$. We say $i$ is *transient* if there exists $j$ such that there is a chain $i \to \cdots \to j$ but no chain with $j \to \cdots \to i$. Non-transient states are called *recurrent*. The following are consequences of simple facts about directed graphs.

(a) The set of recurrent states is partitioned into equivalence classes where $i \sim j$ if there is a chain $i \to \cdots \to j$. On the union of the $\Lambda_i^\pm$ corresponding to the states in each equivalence class is supported exactly one element of $\mathcal{M}_e(T^R)$, which occupies each of these $\Lambda_i^\pm$.

(b) If $i$ is transient, then clearly $\nu(\Lambda_i^\pm) = 0$ for every $\nu \in \mathcal{M}_e(T^R)$. The following claim is a consequence of the structure of $T^R$ (Lemma 8.1) and the fact that for every transient state $j$, there exists a recurrent $k$ such that $j \to \cdots k$.

**Claim 8.2** *$\Lambda_i^\pm$ is the mod 0 union of a collection of pairwise disjoint s-subsets $\{\hat{\Lambda}_{i,\ell}\}_{\ell=1,2,\cdots}$ with the property that for each $\ell$, there exists $n_\ell > 0$ such that $(T^R)^{n_\ell}\hat{\Lambda}_{i,\ell}$ is a u-subset of some recurrent state.*

The discussion in Sects. 8.2.1 and 8.2.2 are summarized as follows:

**Proposition 8.1** *Let $r$ be the number of critical points of $f$. Then there exist ergodic SRB measures*

$$\mu_1,\ \mu_2,\ \cdots,\ \mu_{r'}, \qquad 1 \le r' \le r,$$

*such that for every $C^2(b)$-curve $\gamma$, $m_\gamma$-a.e. $z \in \gamma \cap (\cup \Gamma^s)$ is generic with respect to some $\mu_i$.*

**Proof:** Let $\mathcal{M}_e(T^R) = \{\nu_1, \nu_2, \cdots, \nu_{r''}\}$. Then the $\mu_i$ in this proposition are saturations of the $\nu_j \in \mathcal{M}_e(T^R)$ in the sense of Step 2 in Sect. 8.2.1. Clearly $r' \le r'' \le r$; it may happen that $r' \le r''$ because the saturations of distinct $T^R$-invariant measures may merge. The genericity assertion is proved as follows. If $k$ is a recurrent state, then it is occupied by some $\nu_j$, and hence $m_\gamma$-a.e. $z \in \gamma \cap (\cup \Gamma^s_k)$ is generic with respect to some $\mu_i$. Via Claim 8.2, the same conclusion holds if $k$ is a transient state. $\square$

## 8.3 Accounting for almost every initial condition

It is a fact from general nonuniform hyperbolic theory that if an SRB measure has nonzero Lyapunov exponents, then the set of points generic with respect to it has positive Lebesgue measure. This is a consequence of the absolute continuity of stable foliations [PS]. In general, the set of generic points may not have full Lebesgue measure in any neighborhood of the attractor.

Let $m$ denote the Lebesgue measure on $R_0$.

**Proposition 8.2** *Let $\{\mu_i\}$ be the ergodic SRB measures in Proposition 8.1. Then for $m$-a.e. $z_0 \in R_0$, there exists $\mu_i$ with respect to which $z_0$ is generic, and $z_n \in \cup\Gamma^s$ for some $n > 0$.*

It follows that $\{\mu_i\}$ is the set of all the ergodic SRB measures that $T$ admits. Propositions 8.1 and 8.2 together comprise the proof of Theorem 3.

**Proof:** Let $B$ be the set of points *not* generic with respect to any of the $\mu_i$. We remark that $B$ is a Borel measurable set, for genericity with respect to a given measure is determined by a countable number of test functions. Let $Z^{(k)}$ be as in Sect. 7.3. Let $Y_0 = \{z_0 \in R_0 : z_k \notin Z^{(k)}$ for any $k \ge 0\}$, and for $i \ge 1$, let

$$Y_i = \{z_0 \in R_0 : z_i \in Z^{(i)} \text{ and } z_k \notin Z^{(k)} \text{ for all } k > i\}.$$

Suppose $m(B \cap Y_i) > 0$ for some $i > 0$. Then $m(B \cap T^i Y_i) > 0$, and there is a vertical line $\gamma$ with $m_\gamma(B \cap T^i Y_i) > 0$. Let $\varepsilon > 0$ be a small number. By the Lebesgue density theorem, there exists a short segment $\gamma_0 \subset \gamma$ with the property that $m_\gamma(B \cap T^i Y_i \cap \gamma_0) > (1-\varepsilon) m_\gamma(\gamma_0)$. We will show in the next paragraphs that

points generic with respect to some $\mu_i$ make up a definite fraction of $\gamma_0$, contradicting our choice of $\gamma_0$ if $\varepsilon$ is sufficiently small. (The argument we present also works if $m(B \cap Y_0 \cap \mathcal{C}^{(0)}) > 0$. For the case $m(B \cap Y_0 \cap (R_0 \setminus \mathcal{C}^{(0)})) > 0$, use horizontal instead of vertical lines.)

Let $\tau_0$ denote the tangent vectors to $\gamma_0$, and let $\gamma_j = T^j \gamma_0$. We regard all of $\gamma_0$ (which can be taken to be arbitrarily short) as bound to its nearest critical point, and let $n_1$ be the first time when part of $\gamma_j$ makes a free return to $\mathcal{C}^{(0)}$. As before, let $\Lambda_k = \Lambda_k^+$ or $\Lambda_k^-$. Let $D(\Lambda_k)$ denote the smallest rectangular region bounded by $\Gamma^u$ and $\Gamma^s$-curves that contains $\Lambda_k$. If $\gamma_{n_1}$ crosses some $D(\Lambda_k)$ with two segments of at least comparable lengths extending beyond the two sides of $D(\Lambda_k)$, we consider the segment $\gamma_{n_1} \cap D(\Lambda_k)$ as having reached its final destination and take it out of circulation. We then divide what remains of $\gamma_{n_1}$ into $I_{\mu j}$ and delete those subsegments that do not contain a point of $T^{n_1}(B \cap T^i Y_i)$.

Observe that for $z_0 \in \gamma_0 \cap T^i Y_i$, $(z_0, \tau_0)$ is controlled through time $n_1 - 1$, and by Lemma 7.1, $\tau_{n_1}$ splits correctly (see the proof of Theorem 2(2)). This is true not only for $z_0 \in \gamma_0 \cap T^i Y_i$ but also for $z_0' \in \gamma_0$ such that $z_{n_1}'$ is in the same $I_{\mu j}$ as $z_{n_1}$. We iterate independently each one of the $I_{\mu j}$-segments that are kept. At the next free return we repeat the same procedure, namely we take out subsegments that cross some $D(\Lambda_k)$, divide the rest into $I_{\mu j}$, delete those that do not contain a point in the image of $B \cap T^i Y_i$, and observe that for the remaining segments control is extended to the next free return.

Let $\gamma_0^d = \{z_0 \in \gamma_0 : z_j \text{ is deleted at a free return for some } j > 0\}$, and let $\hat{\gamma}_0 = \{z_0 \in \gamma_0 : z_j \text{ reaches } D(\Lambda_k) \text{ for some } j \text{ and } k \text{ in the required manner}\}$. We note that $m_{\gamma_0}(\gamma_0 \setminus (\gamma_0^d \cup \hat{\gamma}_0)) = 0$. This follows from a sublemma which is the first step in the proof of Lemma 8.4 (see [BY2], Sublemma 4 and its corollary).

Since $(\gamma_0 \cap B \cap T^i Y_i) \cap \gamma_0^d = \emptyset$, we have $(\gamma_0 \cap B \cap T^i Y_i) \subset \hat{\gamma}_0$ mod 0 and that $\hat{\gamma}_0$ is the disjoint union of a countable number of subsegments $\{\omega\}$ with the following properties:

– each $\omega$ is mapped under some $T^{n(\omega)}$ onto a $C^2(b)$-curve that connects two $\Gamma^s$-sides of some $D(\Lambda_k)$;
– $(z_0, \tau_0)$ is controlled up to time $n(\omega)$ for every $z_0 \in \omega$.

From Lemmas 8.2, 8.3 and Proposition 8.1, it follows that there exists $c_1 > 0$ independent of the choice of $\gamma_0$ such that for each $\omega$,

$$m_\gamma\{z_0 \in \omega : z_{n(\omega)} \in \cup\Gamma^s \text{ and is generic w.r.t. some } \mu_k\} > c_1 m_\gamma(\omega).$$

This implies that $m_\gamma\{z_0 \in \gamma_0 : z_0 \text{ is generic w.r.t. some } \mu_k\} > c_1 m_\gamma(\hat{\gamma}_0) > c_1(1-\varepsilon)$ $m_\gamma(\gamma_0)$, contradicting our choice of $\gamma_0$ if $c_1(1-\varepsilon) > \varepsilon$. $\square$

For $i = 1, 2, \cdots, r'$, the set $B_i := \{z \in R_0 : z \text{ is generic with respect to } \mu_i\}$ can be thought of as the **measure-theoretic basin** of $\mu_i$. When there exist multiple SRB measures, the $B_i$'s can be quite delicately interwined (although they are not

"riddled"). We leave it as an exercise for the reader to construct an example of a 1-dimensional map $f$ which when perturbed according to the rules in Sect. 1.1 gives rise to a positive measure set of maps $T$ with the following properties:

(i) $T$ admits $n$ ergodic SRB measures for any $n \geq 2$;

(ii) there is a Cantor set of stable curves that do not meet any $B_i$;

(iii) every open set that meets any one of the curves in (ii) intersects every $B_i$ in a positive Lebesgue measure set.

## 8.4 Correlation decay and Central Limit Theorem

We indicate how Theorem 4 is proved. The setup $T^R : \Lambda \to \Lambda$ is designed so that the statistical properties in question are easily read off from the tail properties of the return time function $R$. To use the results in [Y3] or [Y4] directly, however, we need to consider returns to a single recurrent state. Let $\tilde{\mu}$ be one of the $\mu_j$ in Proposition 8.1, and let $\tilde{\Lambda}$ be one of the $\Lambda_i$ such that $\tilde{\mu}(\Lambda_i) > 0$. For $z \in \tilde{\Lambda}$, we define a return time $\tilde{R}(z)$ of $z$ to $\tilde{\Lambda}$ as follows:

$$\tilde{R}(z) \ = \ t_0 + t_1 + \cdots + t_n,$$

where $t_0 = R(z),\ t_1 = R(T^R(z)),\ \cdots,\ t_n = R((T^R)^n z)$ and $(T^R)^{n+1} z$ is the first return to $\tilde{\Lambda}$ under $T^R$. The results in [Y3] or [Y4] allow us to read off information on the statistical properties of $(T, \tilde{\mu})$ via the asymptotics of $m_{\partial R_0}\{z \in \partial R_0 \cap \tilde{\Lambda} : \tilde{R}(z) > n\}$.

**Lemma 8.5** *There exists $K > 0$ and $\tilde{\theta}_0 < 1$ such that for every $n > 0$,*

$$m_{\partial R_0}\{z \in \partial R_0 \cap \tilde{\Lambda} : \tilde{R}(z) > n\} < K \tilde{\theta}_0^n.$$

This lemma, which we leave as an exercise, is an easy consequence of Lemma 8.4. The results in [Y3] and [Y4] state that if the quantity estimated in Lemma 8.5 is of order $\mathcal{O}(\frac{1}{n^{2+\varepsilon}})$ for some $\varepsilon > 0$, then the Central Limit Theorem holds in the context of Theorem 4. This condition is evidently satisfied here. They also tell us that if this quantity is exponentially small, then every mixing component of $\tilde{\mu}$ has exponential decay of correlations as asserted.

# 9 Global Geometry

## 9.1 Motivation

Nonuniformly hyperbolic attractors have very complicated local structures. The purpose of this section is to develop an understanding of the *coarse geometry* of the attractor $\Omega$ for the maps in question, that is to say, to describe in a finite way the approximate shape and complexity of $\Omega$.

To illustrate the idea of coarse geometry, consider the standard solenoid constructed from $z \mapsto z^2$. A good approximation of the attractor is given by the $k$th forward image of $S^1 \times D_2$, which is a tubular neighborhood of a simple closed curve winding around the solid torus $2^k$ times. For another example, consider piecewise monotonic maps in 1-dimension. Iterates of these maps continue to be piecewise monotonic and can be understood in terms of their monotone pieces.

Returning to the maps under consideration, the standard solenoid example suggests that $R_k$ may be a good approximation of $\Omega$. In analogy with 1-dimension, one may also guess that $R_k$ is a tubular neighborhood of a simple closed curve whose $x$-coordinates vary in a piecewise monotonic fashion. The latter is false, as is evident from the following sequence of pictures: Depicted in (a) is a section of $R_k$ lying between two $C^2(b)$-curves; (b) is the image of (a). As (b) is iterated, the horizontal distance between the tips of the two parabolas increases as shown in (c), until at some point they fall on opposite sides of a component of the critical set, resulting in (d). Since this happens to every "turn" that is created, the geometry of $R_k$ for large $k$ is quite complicated.

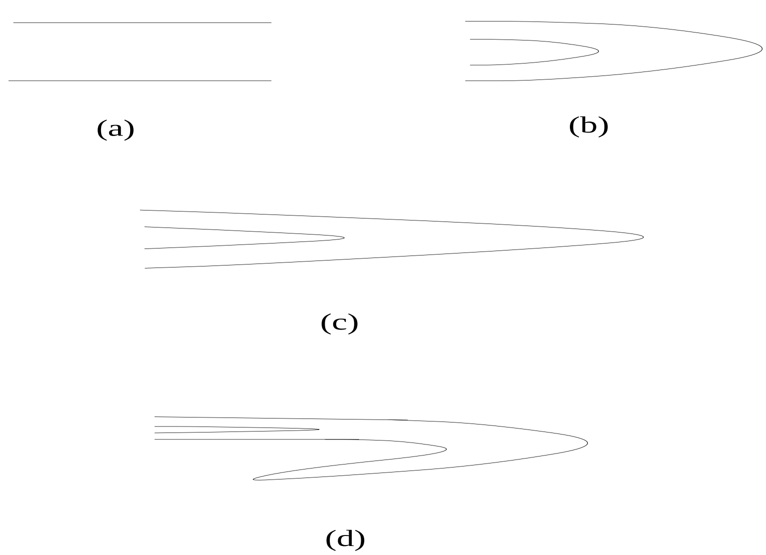



Figure 3 The geometry of $R_k$

The purpose of this section is to introduce the idea of **monotone branches** as basic building blocks for understanding the global structure of $\Omega$. To each map $T$ we will associate a **combinatorial tree** whose edges correspond to monotone branches, and we will show that $\Omega$ has arbitrarily fine neighborhoods made up of unions of finitely many monotone branches. Moreover, the way these branches fit together will tell us exactly how, in finite approximation, $T$ differs from a 1-dimensional map.

## 9.2 Monotone branches

Let $\Gamma$ be the set of critical points as in Sections 3–6. For $z_0 \in R_0$, let $O_+(z_0) = \{z_1, z_2, z_3, \cdots\}$ denote the positive orbit of $z_0$, and write $O_+(\Gamma) = \cup_{z_0 \in \Gamma} O_+(z_0)$.

**Definition 9.1** *Let $\gamma$ be a connected subsegment of $\partial R_k$. We say $\gamma$ is a* **(maximal) monotone segment** *if*

*(i) the two end points of $\gamma$ are in $O_+(\Gamma)$;*

*(ii) $\gamma$ does not intersect $O_+(\Gamma)$ in its interior.*

When we say $\xi_i$ is an end point of a monotone segment, it will be understood that $\xi_0$ is a critical point. We record below some simple facts about monotone segments.

**Lemma 9.1** *Let $\gamma \subset \partial R_k$ be a monotone segment. Then:*

*(a) All points near the two ends of $\gamma$ are in their fold periods; the part of $\gamma$ not in a fold period (respectively bound period), if nonempty, is connected.*

*(b) If part of $\gamma$ is free, then its geometry is as follows: $\gamma$ consists of a relatively long $C^2(b)$-curve connecting two sets of relatively small diameters at the two ends; more precisely, there exists $p$ such that the $C^2(b)$-curve has length $> e^{-\beta p}$ while the diameters of the two small sets are $< b^{\frac{p}{2}}$; also, the curvature of $\partial R_k$ at the end point $\xi_i$ of $\gamma$ is $> b^{-i}$.*

*(c) If $\gamma$ meets $\Gamma$ in $r$ points, $r \geq 0$, then $T(\gamma)$ is the union of $r+1$ monotone segments joined together at the $T$-images of these points.*

**Proof:** (c) follows from the definition of a monotone segment. (a) follows from the way monotone segments are created and from the monotonicity of bound and fold periods (see the proof of Lemma 4.10). The first assertion in (b) follows from estimates on the relative sizes of the parts of $\gamma$ that are in bound versus fold periods; the second follows from the curvature formula in the proof of Lemma 2.4. $\square$

We now begin to study the geometry of certain 2-dimensional objects.

**Definition 9.2** *A simply connected region $S \subset R_k$ is called a* **monotone branch** *if it is bounded by two monotone segments $\gamma$, $\gamma' \subset \partial R_k$ and two ends $E_\xi$ and $E_\zeta$ with the following properties:*

*(i) If the end points of $\gamma$ are $\xi_i$ and $\zeta_j$, then the end points of $\gamma'$ are $\xi'_i$ and $\zeta'_j$ where $\xi_0$ and $\xi'_0$ lie on the upper and lower boundaries of the same component $Q^{(k-i)}$ of*

*$\mathcal{C}^{(k-i)}$, and $\zeta_0$ and $\zeta_0'$ are related in the same way.*

*(ii) $E_\xi = T^i\{z \in Q^{(k-i)}(\xi_0) : |z-\xi_0| < b^{\frac{k-i}{4}}\}$; its* **time of creation** *is said to be $k-i$; $E_\zeta$ and its time of creation are defined analogously.*

*(iii) We define the* **age** *of $E_\xi$ to be $i$ and require that $i < \theta^{-1}(k-i+1)$; there is an analogous limit on the age of $E_\zeta$.*

The definitions of $E_\xi$ and $E_\zeta$ are quite arbitrary, subject only to the following considerations: We want $E_\xi$ to be large enough to contain all the critical orbits that originate from $Q^{(k-i)}(\xi_0)$. On the other hand, we want it to remain relatively small during the life span of the monotone branch, so that the phenomenon depicted in Figure 3 does not occur. We assume $\theta$ is chosen such that for $i < \theta^{-1}(k-i+1)$, $\|DT\|^i b^{\frac{k-i}{4}} < b^{\frac{k-i}{8}} << e^{-\alpha i}$, which is $< d_{\mathcal{C}}(z_i)$ for $z_0 \in \Gamma$ by (IA2) in Section 3; that is to say, if $S$ is a monotone branch of $R_k$, then its ends are at least a certain distance away from $\mathcal{C}^{(k)}$. It is not always easy to visually identify monotone branches, particularly when their boundary segments are in fold periods. When part of $\gamma$ is free, it follows from Lemma 9.1(b) that $S$ consists of a (relatively long) horizontal strip with two small blobs at the two ends.

**Tree structure of a class of monotone branches**

Monotone branches can be constructed as follows. First we declare that $R_0$ is a monotone branch (even though it has no ends). Then if $x_i < x_{i+1}$ are adjacent critical points of the 1-dimensional map $f$, the $T$-image of $\{z = (x, y) : x_i - b^{\frac{1}{4}} < x < x_{i+1} + b^{\frac{1}{4}}\}$ is a monotone branch of $R_1$. In general, let $S$ be a monotone branch of $R_k$. If one of the ends of $S$ is at its maximum allowed age, then $S$ is "discontinued", meaning we do not iterate it further. If not, $T(S)$ is the union of a finite number of monotone branches of $R_{k+1}$. More precisely, if $S \cap \mathcal{C}^{(k)} = \emptyset$, then $T(S)$ is a monotone branch. If $S \cap Q^{(k)} \neq \emptyset$, then $S \supset Q^{(k)}$ (in fact, $S$ extends beyond $Q^{(k)}$ by $> e^{-\alpha k}$ in both directions). If $S$ contains $r$ components of $\mathcal{C}^{(k)}$, then $T(S)$ is the union of $r+1$ monotone branches split roughly along the $T$-images of the middle of each of the $Q^{(k)}$ contained in $S$ (cf. Lemma 9.1(c)).

Let $\mathcal{T} = \cup_k \mathcal{T}_k$ denote the set of all monotone branches inductively constructed this way, with $\mathcal{T}_k$ consisting of branches of $R_k$. More precisely, $\mathcal{T}_0 = \{R_0\}$, and $\mathcal{T}_{k+1}$ is obtained from $\mathcal{T}_k$ via the procedure described above. **We will be working exclusively with monotone branches in $\mathcal{T}$**, which is a proper subset of the set of all monotone branches in Definition 9.2. The set $\mathcal{T}$ has a natural tree structure: we call the branches obtained by mapping forward and subdividing a given branch its **descendants**. Note that every branch in $\mathcal{T}_k$ has a unique **ancestor** in $\mathcal{T}_i$ for every $i < k$, but not all branches in $\mathcal{T}$ have offsprings: the ones with no offsprings are exactly those one of whose ends has reached its maximum allowed age.

We have elected to discontinue a branch before its geometry "deteriorates". An immediate question that arises is what happens to the part of the attractor contained in a discontinued branch. We will show in the next subsection that branches farther

down the tree $\mathcal{T}$ can be used to take its place. We will, in fact, prove the following stronger version of Theorem 5.

**Theorem** 5′ *One can construct special neighborhoods* $\tilde{R}_n$ *as in Theorem 5 using only monotone branches from* $\mathcal{T}_k,\ n \le k < (1 + K\theta)n$.

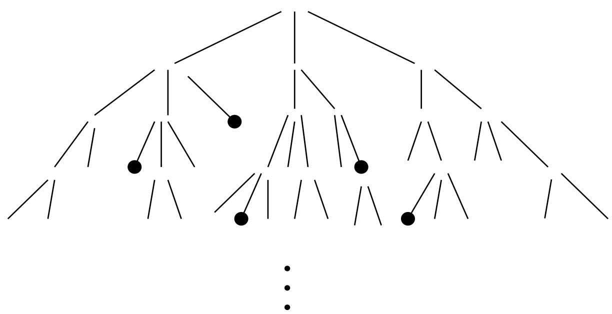


Figure 4 Tree of monotone branches: branches ending in • are discontinued

## 9.3 Replacement of branches

Let $S \in \mathcal{T}_k$ be a branch whose ends are denoted by $E_\xi$ and $E_\zeta$. In the discussion to follow, we assume that $E_\xi$ is fairly advanced in age, meaning $(k - i) \sim \theta i$ where $i$ is the age of $E_\xi$ and $k - i$ is its time of creation. As we search for replacements for $S$, the picture we hope to have is the following. There is a finite collection of branches $\{B\} \subset \cup_{k<j\le(1+K\theta)k}\mathcal{T}_j$ such that

(i) the ends of $B$ are contained in those of $S$; and

(ii) if $S \in \mathcal{S}$ where $\mathcal{S} \subset \mathcal{T}$ is a cover of $\Omega$, then replacing $S$ by $\{B\}$ does not leave any part of $\Omega$ exposed.

Let $Q^{(k-i)}$ be the component of $\mathcal{C}^{(k-i)}$ containing $T^{-i}E_\xi$. We hope to show that $T^{-i}S \subset Q^{(k-i)}$, so that the picture described above pulled back to $Q^{(k-i)}$ is as shown in Figure 5.

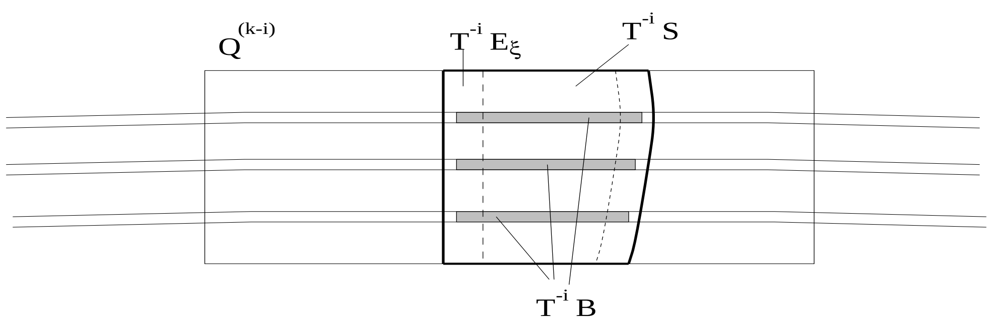



Figure 5 Replacing $S$ by $\{B\}$

We begin to systematically justify this picture. For $j = 0, 1, \cdots, i-1$, let $S_j \in \mathcal{T}_{k-i+j}$ be the ancestor of $S$, so that $S_0$ is the monotone branch of $R_{k-i}$ containing $Q^{(k-i)}$. Let $E_0$ denote the end of $S_0$ contained in $Q^{(k-i)}$, and let $E_j = T^j E_0$. Let the other end of $S_j$ be called $E'_j$. Let $t > k-i$, and let $P \in \mathcal{T}_t$ be such that $P \cap Q^{(k-i)}$ is a horizontal strip bounded by two $C^2(b)$-curves stretching all the way across $Q^{(k-i)}$. We think of $P$ as a *pre-branch* with respect to $S_0$ in the sense that $P \subset S_0$ and it is not yet born when $S_0$ is created. If $P$ is not discontinued, then we let $P_1$ be the (unique) child of $P$ with one end in $E_1$, and assuming $P_1$ is not discontinued, we let $P_2$ be the child of $P_1$ with one end in $E_2$. Similarly, we define $P_3, P_4, \cdots$ up to $P_i$ if it makes sense.

**Lemma 9.2** *There exists $K_1$ depending on $\rho$ such that*
*(i) for all $j$ with $K_1(k-i) < j \le i$, $T^{-j} S_j \subset Q^{(k-i)}$;*
*(ii) if $P_j$ is defined for all $j \le K_1(k-i)$, then it is defined for all $j \le i$; moreover, for each $j \ge K_1(k-i)$, $P_j \subset S_j$, and the two ends of $P_j$ are contained in the two ends of $S_j$.*

We isolate the following sublemma, the ideas in which are also used elsewhere. See Sect. 6.1 for notation.

**Sublemma 9.1** *Let one of the horizontal boundaries of $Q^{(s)}$, any $s$, be identified with $[-\rho^s, \rho^s]$, with the critical point corresponding to 0. Then for every $I_{\mu_0 j_0} \subset [-\rho^s, \rho^s]$, there exists $n < K|\mu_0|$ such that $T^n I_{\mu_0 j_0}$ traverses completely a component of $\mathcal{C}^{(0)}$.*

**Proof:** Let $\omega_0 = I_{\mu_0 j_0}$, and let $r_0$ be the first time when part of $\omega_0$ makes a free return with $T^{r_0}\omega_0$ containing an $I_{\mu j}$ of full length. By Corollary 4.3, either $T^{r_0}\omega_0$ contains one of the outermost $I_{\mu j}$ (which we will call $\tilde{I}$) or it contains some $I_{\mu_1 j_1}$ with $|\mu_1| < K\beta|\mu_0|$. In the latter case, we let $\omega_1 = I_{\mu_1 j_1}$ and continue to iterate until $r_1$ iterates later when part of $T^{r_1}\omega_1$ is free and contains either $\tilde{I}$ or some $I_{\mu_2 j_2}$ with $|\mu_2| < K\beta|\mu_1|$. After a finite number of iterates, we have $T^{r_q} w_q \supset \tilde{I}$.

From Corollary 4.3, we see that at the end of its bound period, $T^p \tilde{I}$ has length $>> \delta$. Inductively define $\tilde{I}_{p+j} = T(\tilde{I}_{p+j-1}) \setminus \mathcal{C}^{(0)}$ for $j = 1, 2, \cdots$. Then $\tilde{I}_{p+j}$ is a connected $C^2(b)$-curve which grows essentially exponentially – until it crosses completely a component of $\mathcal{C}^{(0)}$. Since $r_i \sim |\mu_i|$ up to the point when $T^{r_q} w_q \supset \tilde{I}$, and the growth is exponential thereafter, we conclude that the end game is reached in a total of $< K|\mu_0|$ iterates. $\square$

**Proof of Lemma 9.2:**

**Claim 9.1** *There exists $K_1$ (depending on $\rho$) such that $T^{-K_1(k-i)} S_{K_1(k-i)} \subset Q^{(k-i)}$.*

*Proof of Claim 9.1:* We identify the upper horizontal boundary of $Q^{(k-i)}$ with the interval $[-\rho^{k-i}, \rho^{k-i}]$, with the critical point corresponding to 0, and let $n_1$ be the

smallest $n$ such that $T^n[0,\frac{1}{2}\rho^{k-i}]$ intersects the horizontal boundary of some $Q^{(k-i+n)}$. From Sublemma 9.1, $n_1 < K_1(k-i)$ for some $K_1 = K(\rho)$. The claim is proved once we show that $T^{-(n_1+1)}S_{n_1+1} \subset Q^{(k-i)}$. Let $[0,\ell]$ be the shortest interval such that $T^{n_1}[0,\ell]$ contains the entire horizontal boundary of a $Q^{(k-i+n_1)}$. Since this boundary is free, $\ell < \frac{1}{2}\rho^{k-i} + e^{-c'n_1}\rho^{k-i+n_1}$, which is $\approx \frac{1}{2}\rho^{k-i}$. Let $\hat{S}_{n_1}$ be the section of $R_{k-i+n_1}$ from $E_{n_1}$ to the middle of $Q^{(k-i+n_1)}$. Since $b^{\frac{k-i}{4}}K^{K_1(k-i)} << \rho^{k-i+K_1(k-i)}$, we have that $T^{n_1}Q^{(k-i)} \supset \hat{S}_{n_1}$. It remains to show $S_{n_1+1} = T(\hat{S}_{n_1})$, for which we need only to check that $T(\hat{S}_{n_1})$ is a monotone branch. To do that, it suffices to show that for $j < n_1$, $T^j[0,\ell]$ does not contain the horizontal boundary of any $Q^{(k-i+j)}$. Suppose it does for some $j$. By our choice of $n_1$, this can happen only if $\ell > \frac{1}{2}\rho^{k-i}$ and $\mid T^j[\frac{1}{2}\rho^{k-i},\ell] \mid \geq \rho^{k-i+j}$, which is impossible, for $\mid T^j[\frac{1}{2}\rho^{k-i},\ell] \mid < e^{-c'(n_1-j)}\rho^{k-i+n_1}$. $\diamondsuit$

Suppose we are guaranteed that $P_{n_1}$ exists. We show next that $P_{n_1+1}$ exists and has the properties in Lemma 9.2(ii). Let $\gamma$ be the part of a horizontal boundary of $P$ that lies below $[0,\ell]$. From the estimates above, we know that $T^{n_1}\gamma$ is $C^0$ very near $T^{n_1}[0,\ell]$. Let $\hat{P}_{n_1}$ be the section of $T^{n_1}(P \cap Q^{(k-i)})$ that runs from $E_{n_1}$ to the middle of some $Q^{(t+n_1)} \subset Q^{(k-i+n_1)}$. We claim that $P_{n_1+1} = T(\hat{P}_{n_1})$. Clearly, $P_{n_1+1} \subset S_{n_1+1}$. To see that $P_{n_1+1}$ is a monotone branch, it suffices to observe that for $j < n_1$, $T^{-n_1+j}\hat{P}_{n_1} \cap \mathcal{C}^{(t+j)} = \emptyset$, which is an immediate consequence of the fact that $T^{-n_1+j}\hat{S}_{n_1} \cap \mathcal{C}^{(k-i+j)} = \emptyset$.

We are now ready to show that $P_j$ exists for all $j \leq i$. Suppose that $P_{j-1}$ exists. The only reason why $P_j$ may not exist is that one of its ends has reached its maximum allowed age. Of the two ends of $P_{n_1+1}$, the one contained in $E_{n_1+1}$ is clearly created earlier, which means that of the two ends of $P_{j-1}$, the one contained in $E_{j-1}$ is created earlier. It suffices therefore to check that this end survives the step from $P_{j-1}$ to $P_j$. It does, because it is created later than $E_{j-1}$ and has the same age as $E_{j-1}$, and, by definition, $E_{j-1}$ has not reached its maximum allowed age.

From here on we argue inductively that the relations in Lemma 9.2(ii) between $P_j$ and $S_j$ hold from $j = n_1+2$ to $j = i$. Assume this is true for $j-1$, and that $S_{j-1}$ has more than one child. Then $S_j = T(\hat{S}_{j-1})$ where $\hat{S}_{j-1}$ is the section of $S_{j-1}$ from $E_{j-1}$ to the middle of some $Q^{(k-i+j-1)}$. Since by inductive assumption $P_{j-1}$ has its ends contained in those of $S_{j-1}$, we are assured that it traverses some $Q^{(t+j-1)} \subset Q^{(k-i+j-1)}$. Letting $\hat{P}_{j-1}$ be the section of $P_{j-1}$ from its end in $E_{j-1}$ to the middle of $Q^{(t+j-1)}$, we see that $P_j = T(\hat{P}_{j-1})$ has the desired properties.

This completes the proof of Lemma 9.2. $\square$

**Proof of Theorem** $5'$**:** Let $\mathcal{S}_0 = \{R_0\}$, and assume that for each $n \leq m$, a collection of monotone branches $\mathcal{S}_n$ is selected so that $\tilde{R}_n := \cup_{S\in\mathcal{S}_n} S$ is a neighborhood of the attractor, and each $S \in \mathcal{S}_n$ has the following properties:

(i) $S \in \mathcal{T}_k$ for some $n \leq k \leq (1+3\theta)n$;

(ii) if an end of $S$ is of age $i$, i.e. it is created at time $k-i$, then $2\theta i \leq k-i+1$.

Note that (ii) is a more stringent requirement than the definition of monotone branches.

The collection $\mathcal{S}_{m+1}$ is defined as follows. For each $S \in \mathcal{S}_m$, if the ends of $S$ have not reached their maximum ages as allowed by (ii) above, then we put the children of $S$ in $\mathcal{S}_{m+1}$. If one of its ends has reached this age, then we choose a collection of branches $\{P\}$ to be specified in the next paragraph, construct from each $P$ a monotone branch $P_i$ as in Lemma 9.2, replace $S$ by $\{P_i\}$ and put the children of $P_i$ in $\mathcal{S}_{m+1}$.

Suppose for definiteness that $S \in \mathcal{T}_k$, and its end $E$ has reached age $i$ where

$$2\theta i = k - i + 1. \tag{13}$$

Let $Q^{(k-i)}$ be the component of $\mathcal{C}^{(k-i)}$ containing $T^{-i}E$. Let $\{P\}$ be the subcollection of $\mathcal{S}_{k-i+1}$ with the property that $P \cap Q^{(k-i)} \neq \emptyset$. Observe immediately that by our inductive hypotheses, $P$ is a monotone branch of $R_{\tilde{k}}$ for some $\tilde{k}$ with

$$k - i + 1 \ \leq\ \tilde{k} \ \leq\ (1+3\theta)(k-i+1). \tag{14}$$

Since $e^{-\alpha(1+3\theta)(k-i+1)} >> \rho^{k-i}$, it follows that $P$ intersects $Q^{(k-i)}$ in a horizontal strip bounded by $C^2(b)$ curves. Note also that since the union of the elements of $\mathcal{S}_{k-i+1}$ covers $\Omega$, we have $\cup P \supset (Q^{(k-i)} \cap \Omega)$.

To justify the validity of this replacement procedure, we need to show that

(a) for each $P$ as above, $P_{K_1(k-i)}$ is well defined where $K_1$ is as in Lemma 9.2;
(b) $P_i$ is a monotone branch of $R_j$ for some $j \leq (1+3\theta)m$.

Suppose that an end of $P$, which is a branch of $R_{\tilde{k}}$, is of age $\tilde{i}$. Then

$$2\theta\tilde{i} \ \leq \tilde{k} - \tilde{i} + 1. \tag{15}$$

To prove (a), if suffices to verify that this end lasts another $K_1(k-i)$ iterates, i.e.

$$\theta\ [\tilde{i} + K_1(k-i)] \ \leq\ \tilde{k} - \tilde{i} + 1.$$

This is true because $\theta\tilde{i} \leq \frac{1}{2}(\tilde{k} - \tilde{i} + 1)$ by (15), and

$$\begin{aligned} K_1\theta(k-i) \ &\leq\ K_1\theta(\tilde{k}+1) \ = \ K_1\theta[(\tilde{k} - \tilde{i} + 1) + \tilde{i}] \\ &\leq\ K_1\theta(\tilde{k} - \tilde{i} + 1)(1 + \frac{1}{2\theta}) \ << \ \frac{1}{2}(\tilde{k} - \tilde{i} + 1). \end{aligned}$$

The first inequality above is by (14) and the second by (15).

To prove (b), we need to check that the age of the end of $P_i$ that is contained in $E$, namely $\tilde{k} + i$, is $\leq (1+3\theta)m$. Observe first that $i \leq m$. This is because the replacement procedure described in Lemma 9.2 does not change the ages of the respective ends of the monotone branch in question. (The age of an end is equal to the "age" of the critical orbits it contains.) Thus it remains to check that

$$\tilde{k} \ \leq\ (1+3\theta)(k-i+1) \ = \ (1+3\theta)2\theta i \ < \ 3\theta i \ \leq\ 3\theta m,$$

the first inequality above coming from (14) and the equality from (13). This completes the proof of Theorem $5'$. $\square$

We mention two bonuses of this construction.

First, it can be seen inductively that for every $S \in \mathcal{S}_n$, if $S$ is a branch of $R_k$, then the two monotone segments of $\partial R_k$ that bound $S$ must necessarily be from different components of $\partial R_0$. This is used in Sect. 10.6.

Second, we claim that if $\deg(f) \neq 0$, then all of our monotone branches $S \in \mathcal{S}_m$ intersect the attractor $\Omega$ in an essential way. Let us call a monotone branch $S$ **essential** if every curve connecting the two monotone segments $\gamma$ and $\gamma'$ in $\partial S$ meets $\Omega$. Observe first that $R_0 \in \mathcal{S}_0$ is essential if $\deg(f) \neq 0$. If not, then there exists a curve $\omega$ connecting the two components of $\partial R_0$ that does not meet $\Omega$. Since $\Omega = \cap_k R_k$, this implies that for some $k$, $R_k \cap \omega = \emptyset$, which is absurd since $R_k$ is not contractible. Assuming that $S \in \mathcal{S}_m$ is essential, then clearly all the monotone branches that comprise $T(S)$ are essential if no end replacements are needed in the next step. If an end replacement is required, then since the new branches are the images of parts of earlier essential branches, they are again essential.

## 9.4 The coarse geometry of $\Omega$

We explain in the following sequence of pictures exactly how, in finite approximation, the geometry of $\Omega$ differs from that of a small tubular neighborhood of a single curve. These pictures are justified by Lemma 9.2. Referring back to Figure 3(c), we may think of the region between the parabolas as made up to two ends belonging to adjacent branches. We know from Lemma 9.2 that long before the tips of these parabolas "separate", that is, before the ends in question reach their maximum allowed age, there are *pre-branches* inside running parallel to these parabolas. In Figure 6 below, the pre-branches are shown in grey, and the zig-zagging cut-lines represent pre-images of the critical set. These cut-lines will become "turns" before the ends in question reach their maximum allowed age.

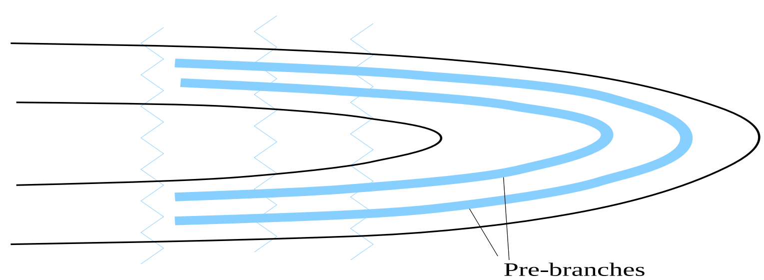
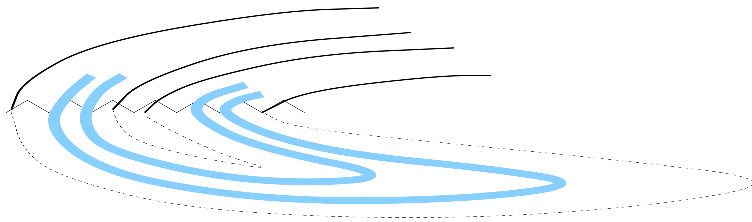


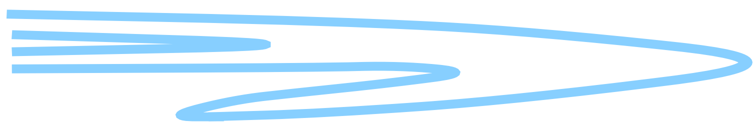

Figure 6 Pre-branches waiting to be released

As this age is reached, the pre-branches are released. Figure 7(a) shows four newly released montone branches grafted onto a branch created earlier. Once released, the

new branches evolve independently, resulting possibly in the configuration in Figure 7(b) (cf. Figure 3(d)).



**(a)**



**(b)**

Figure 7 Newly released monotone branches evolving independently

The boundaries of every turn (or pair of ends) created every step of the way will in time separate, releasing new branches grafted onto ones born earlier. As the new branches evolve, they create new turns, which again will last for only a finite duration of time. In terms of global geometry, this, in a sense, is the *only* way in which $T$ differs from a 1-dimensional map. Tip replacements are scheduled to take place roughly once every $\sim \log \frac{1}{b}$ iterates, so that in the limit as $b$ tends to 0, no replacement is needed – as it should be for 1-dimensional maps.

# 10 Symbolic Dynamics and Topological Entropy

The goals of this section are (1) to introduce a natural and unambiguous coding of all points on the attractor $\Omega$ for the maps in question, and (2) to use this coding to obtain results on topological entropy and equilibrium states.

## 10.1 Coding of points on the attractor

Abusing notation slightly, let $x_1 < x_2 < \cdots < x_r < x_{r+1} = x_1$ be the critical points of $f$ in the order in which they appear on the circle, and let $\mathcal{C}_i := \mathcal{C} \cap \mathcal{C}_i^{(0)}$ where $\mathcal{C}_i^{(0)}$ is the component of $\mathcal{C}^{(0)}$ containing $x_i$. We remark that $\mathcal{C}_i$ may be a fractal set, and

that for an arbitrary $z \in R_0$ near $\mathcal{C}_i$, it does not always make sense to think of $z$ as being located on the left or on the right of $\mathcal{C}_i$. The goal of this subsection is to show that points on $\Omega$ are special, in that for them this left/right notion is always well defined.

Recall that if $Q^{(k)}$ is a component of $\mathcal{C}^{(k)}$, then $\hat{Q}^{(k)}$ is the component of $R_k \cap \mathcal{C}^{(k-1)}$ containing $Q^{(k)}$. In particular, $\hat{Q}^{(k)} \setminus Q^{(k)}$ has a left and a right component.

**Lemma 10.1** *The critical set $\mathcal{C}$ partitions $\Omega \setminus \mathcal{C}$ into disjoint sets $A_1, \cdots, A_r$ as follows:*

- *For $z = (x, y) \notin \mathcal{C}^{(0)}$, $z \in A_i$ if and only if $x_i < x < x_{i+1}$.*
- *For $z \in \mathcal{C}_i^{(0)} \setminus \mathcal{C}_i$, let $Q^{(k)}$ be such that $z \in \hat{Q}^{(k)} \setminus Q^{(k)}$. Then $z \in A_i$ if it lies in the right component of $\hat{Q}^{(k)} \setminus Q^{(k)}$; $z \in A_{i-1}$ if it lies in the left component of $\hat{Q}^{(k)} \setminus Q^{(k)}$.*

**Proof:** This lemma is an immediate consequence of our description of critical regions (Theorem 1(1)). The sets $\{A_i\}$ are defined by the conditions above. What sets points in $\Omega$ apart from arbitrary points in $R_0$ is that $z \in \Omega$ implies $z \in R_k$ for all $k$, so that for $z \in \mathcal{C}^{(0)}$, there are only two possibilities: either $z \in \cap_{k \geq 0} \mathcal{C}^{(k)}$, in which case it is a critical point, or there is a largest $k$ such that $z \in \mathcal{C}^{(k-1)}$. In the latter case, it follows from the geometric relation between $\mathcal{C}^{(k)}$ and $\mathcal{C}^{(k-1)}$ that $z \in \hat{Q}^{(k)} \setminus Q^{(k)}$ for some $Q^{(k)}$. $\square$

Lemma 10.1 gives a well defined **address** $a(z)$ for all $z \in \Omega \setminus \mathcal{C}$. We write $a(z) = i$ if $z \in A_i$. Points in $\mathcal{C}$ have two addresses; for example, for $z \in \mathcal{C}_i$, $a(z) =$ both $i-1$ and $i$. This in turn allows us to attach to each $z_0 \in \Omega$ with $z_i \notin \mathcal{C}$ for all $i$ an **itinerary** $\iota(z_0) = (\cdots, a_{-1}, a_0, a_1, \cdots)$ where $a_i = a(z_i)$. Orbits that pass through $\mathcal{C}$ have exactly two itineraries as $T^i\mathcal{C} \cap \mathcal{C} = \emptyset$ for all $i$.

We would like to show that the symbol sequence $\iota(z_0)$ uniquely determines $z_0$. This may fail in a trivial way: Let $I_i = [x_i, x_{i+1}]$. Then our coding is clearly not unique if for some $i$, $f(I_i)$ wraps all the way around the circle, meeting some $I_j$ more than once. For simplicity of exposition *we will assume this does not happen.* If it does, it suffices to consider the partition on $\Omega$ whose elements correspond to the connected components of $I_i \cap f^{-1} I_j$.

## 10.2 Coding of monotone branches

**Coding of monotone segments of $\partial R_k$.** Observe that points in $\partial R_k$ also have well-defined $a$-addresses in the spirit of Lemma 10.1: if $z \in \partial R_k \cap \mathcal{C}^{(k)}$, then its location with respect to $\Gamma_k$ is obvious (except when $z \in \Gamma_k$). This allows us to assign in a unique way a $k$-block $[a_{-k}, \cdots, a_{-1}]$ to each monotone segment $\gamma$ of $\partial R_k$. We write $\iota(\gamma) = [a_{-k}, \cdots, a_{-1}]$.

**Coding of monotone branches of $R_k$.** Each $S \in \mathcal{T}_k$, $k > 0$, is associated with a block $\iota(S) = [a_{-k}, \cdots, a_{-1}]$ defined inductively as follows: Let $S \in \mathcal{T}_{k-1}$ be such that

$\iota(S) = [a_{-(k-1)}, \cdots, a_{-1}]$. If $S \cap \mathcal{C}^{(k-1)} = \emptyset$, then it lies between two components of $\mathcal{C}$, say $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$, and $\iota(T(S)) := [a'_{-k}, \cdots, a'_{-1}]$ where $a'_{-1} = i$ and $a'_{-j} = a_{-j+1}$ for $j > 1$. If $S \cap \mathcal{C}^{(k-1)} \neq \emptyset$, then $S = \hat{S}_1 \cup \cdots \cup \hat{S}_n$ where $\hat{S}_1$ is the section of $S$ from one end to the middle of the first $Q^{(k-1)}$ that it meets, $\hat{S}_2$ is the section from the middle of this $Q^{(k-1)}$ to the middle of the next component of $\mathcal{C}^{(k-1)}$ etc., and the $a'_{-1}$-entry of $\iota(T(\hat{S}_j))$ is defined according to the location of $\hat{S}_j$. Note that this coding of branches in $\mathcal{T}$ is injective, i.e. $S \neq S'$ implies $\iota(S) \neq \iota(S')$, and that if $\gamma$ and $\gamma'$ are monotone segments that bound $S$, then $\iota(\gamma) = \iota(\gamma') = \iota(S)$. Note also that the replacement procedure in Sect. 9.3 corresponds to replacing $[a_{-k}, \cdots, a_{-1}]$ by blocks of the form $[*, \cdots, *, a_{-k}, \cdots, a_{-1}]$.

**Coding of arbitrary points in $R_0$.** For points in certain locations of $R_0$, there is no meaningful way of assigning to it an address as we did in Sect. 10.1. Instead, for each $k \geq 0$, we define the $\tilde{a}^{(k)}$-address(es) of $z \in R_k$ as follows: $\tilde{a}^{(k)}(z)$ has the obvious definition if $z \notin \mathcal{C}^{(k)}$; if $z = (x, y) \in Q^{(k)}$ for some $Q^{(k)} \subset \mathcal{C}_i^{(0)}$, we let $\tilde{a}^{(k)}(z) = i$ if $x > \hat{x} - b^{\frac{k}{4}}$ where $\hat{z} = (\hat{x}, \hat{y})$ is one of the critical points in $\partial Q^{(k)}$; $\tilde{a}^{(k)}(z) = i - 1$ if $x < \hat{x} + b^{\frac{k}{4}}$. Clearly $\tilde{a}^{(k)}$-addresses are not unique: an open set of points in the middle part of each $Q^{(k)} \subset \mathcal{C}_i^{(0)}$ have as their $\tilde{a}^{(k)}$-addresses both $i - 1$ and $i$.

We further introduce the following notation:

$$\pi_\Omega([a_n, a_{n+1}, \cdots, a_m]) = \{z_0 \in \Omega : a(z_i) = a_i,\ n \leq i \leq m\};$$

$$\pi_{R_0}([a_{-k}, a_{-k+1}, \cdots, a_{-1}]) = \{z_0 \in R_k : \tilde{a}^{(k-i)}(z_{-i}) = a_{-i},\ 1 \leq i \leq k\};$$

"$\tilde{a}^{(k-i)}(z_{-i}) = a_{-i}$" above means $a_{-i}$ is an admissible $\tilde{a}^{(k-i)}$-address of $z_{-i}$.

**Lemma 10.2** *(i) Every $S \in \mathcal{T}_k$, $k \geq 1$, is $= \pi_{R_0}(\iota(S))$ and contains a neighborhood of $\pi_\Omega(\iota(S))$.*
*(ii) Given $z_0 \in \Omega$ and $n \in \mathbb{Z}^+$, there exists $k$ with $n \leq k \leq n(1+3\theta)$ and $S = S(z_0, n) \in \mathcal{T}_k$ such that $z_0 \in \pi_\Omega(\iota(S))$.*

**Proof:** That $S = \pi_{R_0}(\iota(S))$ follows inductively from the definitions of these two objects. That $S$ contains a neighborhood of $\pi_\Omega(\iota(S))$ is also obvious inductively. For (ii), we know from Theorem 5′ that there exists $S \in \mathcal{T}_n$ with $z_0 \in S$. The only way one can have $z_0 \notin \pi_\Omega(\iota(S))$ is that at the time $S$ is created, say at time $k - i$, $T^{-i}S$ meets the mid $b^{\frac{k-i}{4}}$-section $E$ of some $Q^{(k-i)}$ and extends to the left of $E$, while $z_{-i} \in E$ and lies to the "right" of $\Gamma \cap Q^{(k-i)}$ in the sense of Lemma 10.1. Let $S_0$ be the ancestor of $S$ in $\mathcal{T}_{k-i}$, and let $S_1$ be the descendant of $S_0$ that contains the right half of $Q^{(k-i)}$. Our replacement procedure guarantees that there exists $S' \in \mathcal{T}$ that is either a descendant of $S_1$ or a replacement for a descendent of $S_1$ which contains $z_0$. □

Let

$$\Sigma \ := \ \{\mathbf{a} = (a_i)_{i=-\infty}^{\infty} : \iota(z_0) = \mathbf{a} \text{ for some } z_0 \in \Omega\},$$

and let $(\sigma\mathbf{a})_i = (\mathbf{a})_{i+1}$ denote the shift operator. It is easy to check that $\Sigma$ is a closed subset of $\Pi_{-\infty}^{\infty}\{1, 2, \cdots, r\}$ with $\sigma^{-1}\Sigma \subset \Sigma$. Extending our definition of $\pi_\Omega$ to infinite sequences and writing $\pi = \pi_\Omega$, we have that $\pi(\mathbf{a})$ is the set of all points $z_0 \in \Omega$ with $\iota(z_0) = \mathbf{a}$. The following proposition, whose proof occupies all of the next subsection, completes the proof of Theorem 6.

**Proposition 10.1** *For every* $\mathbf{a} \in \Sigma$, $\pi(\mathbf{a})$ *consists of exactly one point, and* $\pi : \Sigma \to \Omega$ *is a continuous mapping.*

Let $B(z_0, \varepsilon)$ denote the ball of radius $\varepsilon$ centered at $z_0$, and let us say $S \in \mathcal{T}_k$ is *compatible* with $\mathbf{a} = (a_i)$ if $\iota(S) = [a_{-k}, \cdots, a_{-1}]$. Proposition 10.1 follows immediately from Lemma 10.2(i) and Proposition 10.1$'$ below.

**Proposition 10.1$'$** *Given* $\mathbf{a} \in \Sigma$, $z_0 \in \pi(\mathbf{a})$, *and* $\varepsilon > 0$, *there exists* $S \in \mathcal{T}_{n+m}$ *compatible with* $\sigma^n\mathbf{a}$ *such that* $T^{-n}S \subset B(z_0, \varepsilon)$.

## 10.3 Uniqueness of point in $\Omega$ corresponding to each itinerary

We begin with a situation that resembles that in 1-dimension.

**Lemma 10.3** *Let* $\mathbf{a}, z_0$ *and* $\varepsilon$ *be as in Proposition* 10.1$'$. *Suppose that for some* $k$, *the component of* $R_k \cap B(z_0, \varepsilon)$ *containing* $z_0$, *which we denote by* $H$, *is bounded by two* $C^2(b)$ *subsegments* $\gamma$ *and* $\gamma'$ *of* $\partial R_k$ *cutting across* $B(z_0, \varepsilon)$ *as shown with*

$$\text{Hausdorff distance } (\gamma, \gamma') < \varepsilon^{10}.$$

*Then there exists* $S \in \mathcal{T}_{n+m}$ *compatible with* $\sigma^n\mathbf{a}$ *such that* $T^{-n}S \subset H$.

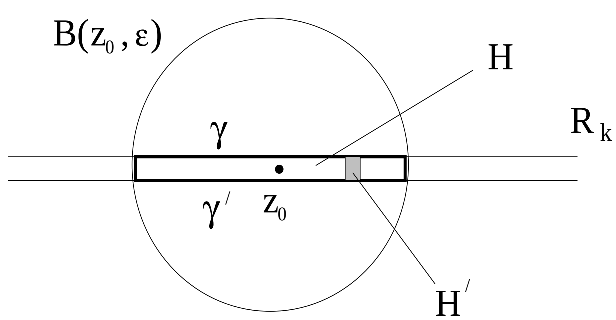



Figure 8 The situation considered in Lemma 10.3

**Proof:** Our plan of proof is as follows. Since $z_0 \in \Omega$, we have, for $i = 1, 2, \cdots$, a monotone branch $S_i \in \mathcal{T}_{k+n_i}$, $n_i > i$, such that $z_i \in S_i$. Then $T^{-i}S_i \subset R_k$ for all $i$. We need to show that for $i$ sufficiently large, $T^{-i}S_i \subset H$. To prevent $T^{-i}S_i$ from exiting

$B(z_0, \varepsilon)$ via the right boundary of $H$, we will show that for some section $H' \subset H$ as shown and $k' > 0$, $T^{k'}(H')$ is a component of $\mathcal{C}^{(k+k')}$, so that the left and right boundaries of $H'$ have incompatible $\tilde{a}^{(k+k')}$-addresses. Assuming $i > k'$, it will follow (using Lemma 10.2) that $T^{-i}S_i$ cannot meet both the left and right boundaries of $H'$. Being connected and contained in $R_k$, $T^{-i}S_i$ must meet both boundaries of $H'$ in order to exit $B(z_0, \varepsilon)$ from the right. The left boundary of $H$ is blocked off similarly.

The proof that $T^{k'}H$ crosses a component of $\mathcal{C}^{(k+k')}$ for some $k'$ is similar to that of Sublemma 9.1, but there are two differences: initially at least, we do not know the lengths of $T^j\gamma$ relative to their distances to the critical set, and we must control the shearing between $\gamma$ and $\gamma'$ as we iterate. Details of the proof follow.

Consider first the case where $z_0 \notin \mathcal{C}^{(0)}$. Let $\gamma_0$ be a subsegment of $\gamma$ of length $\frac{\varepsilon}{2}$ located half-way between $z_0$ and the right boundary of $H$. We first describe how to locate $\gamma_0 \cap H'$. Let $n_1$ be the first time when $T^i(\gamma_0)$ meets $\mathcal{C}^{(0)}$. If $T^{n_1}\gamma_0$ contains an $I_{\mu j}$ of full length, then we let $\gamma_1 \subset T^{n_1}\gamma_0$ correspond to the longest $I_{\mu j}$ or segment outside of $\mathcal{C}^{(0)}$, whichever is longer. If not, we let $\gamma_1 = T^{n_1}\gamma_0$. In both cases, we let $n_2$ be the first time when part of $T^{n_2-n_1}\gamma_1$ makes a free return. Choose $\gamma_2 \subset T^{n_2-n_1}\gamma_1$ as before, let $n_3$ be the first time when part of $T^{n_3-n_2}\gamma_2$ makes a free return, and so on. Using the fact that $\partial R_n$ is controlled (Proposition 5.1), we see that the $\gamma_i$ increase in length, so that there exists some $i_0$ such that $\gamma_{i_0}$ contains an $I_{\mu j}$. From then on, the argument in Sublemma 9.1 produces an $i_1$ such that $T^{n_{i_1}-n_{i_1-1}}\gamma_{n_{i_1-1}}$ traverses a component of $\mathcal{C}^{(0)}$.

We now proceed to construct $H'$. Letting $\tau_0$ denote unit tangent vectors to $\gamma$, we have that $\|DT^i(\xi_0)\tau_0\| \geq c > 0$ for all $\xi_0 \in \gamma_0$ and $i \leq n_1$. Through each $\xi_0 \in \gamma_0$, therefore, is a stable curve of order $n_1$ connecting $\xi_0$ to a point in $\gamma'$ less than $\varepsilon^9$ away (see Sect. 2.2 and Lemma 2.9). Let $H_0$ be the region between $\gamma$ and $\gamma'$ made up of the union of these stable curves.

Since we do not know how close $T^{n_1}\gamma_0$ gets to the critical set, we cannot continue to claim the expanding property of $\tau_0$ beyond time $n_1$. Instead, we observe that for $\xi_0 \in \gamma_1$, $\|DT^j(\xi_0)\binom{0}{1}\| \geq 1$ for $j \leq n_2 - n_1$ so that through each $\xi_0 \in \gamma_1$, there is a stable curve of order $n_2 - n_1$. Assuming that these stable curves meet $T^{n_1}\gamma_0'$, we define $H_1$ to be the region between $T^{n_1}\gamma_0$ and $T^{n_1}\gamma_0'$ spanned by these curves, and check that $H_1$ can be chosen to be a subregion of $T^{n_1}H_0$.

To justify the last sentence, observe first that if $\gamma_1 \subset I_{\mu_1 j}$, then $e^{-\mu_1} > \varepsilon$. This is true regardless of whether $\gamma_1 = T^{n_1}\gamma_0$. Second, since the contractive field $e_{n_2-n_1}$ near $\gamma_1$ makes angles $\sim e^{-\mu_1}$ with $T^{n_1}\gamma_0$ and with $T^{n_1}\gamma_0'$, every point in $\gamma_1$ is connected by a stable curve to a point in $T^{n_1}\gamma_0'$ not more than a distance of $(b^{n_1}\varepsilon^9)/e^{-\mu_1} < b^{n_1}\varepsilon^8 << e^{-\mu_1}$ away. This allows us to define $H_1$. Finally, we may need to trim the edges of $H_1$ by a length $\sim b^{n_1}\varepsilon^8$ in order to fit it inside $T^{n_1}H_0$. This is easily done since $|\gamma_1| > \min(\varepsilon, \frac{1}{\mu_1^2}e^{-\mu_1})$.

At time $n_2$, we again do not know how close $T^{n_2-n_1}\gamma_1$ is to the critical set, and so we use $\|DT^j\binom{0}{1}\| \geq 1$ for $j \leq n_3 - n_2$ to construct new stable curves which are then

used to construct $H_2$. Observe that compared to time $n_1$, the situation has improved: $|\gamma_2| \geq |\gamma_1|$, and the segments $\gamma_2$ and $\gamma_2'$ are closer than before. We construct $H_3$, $H_4$, $\cdots$, until time $n_i$, when $T^{n_{i_1}-n_{i_1-1}} H_{i_1-1} \supset Q$, a component of $\mathcal{C}^{(k+n_{i_1})}$. Letting $k' = n_{i_1}$ and $H' = T^{-n_{i_1}}(Q)$, the proof for the case $z_0 \notin \mathcal{C}^{(0)}$ is complete.

For $z_0 \in \mathcal{C}^{(0)} \setminus \mathcal{C}$, let $j$ be such that $z_0 \in \hat{Q}^{(j)} \setminus Q^{(j)}$. If $k$ in the statement of the lemma is $\geq j$, repeat the argument above with $n_1 = 0$. If not, replace $k$ by $j$ and $\varepsilon$ by $\min(\varepsilon, \frac{1}{2}\rho^j)$ and let $n_1 = 0$. The case of $z_0 \in \mathcal{C}$ is dealt with similarly. $\square$

Recall that for all $z_0 \in \Omega$, at every return to $\mathcal{C}^{(0)}$, $z_i$ is h-related, and bound and fold periods are well defined. (See Section 3 for definitions.)

**Proof of Proposition** $10.1'$ : Let $\mathbf{a} \in \Sigma$ and $z_0 \in \pi(\mathbf{a})$ be given. We wish to arrange for the scenario in Lemma 10.3 at $z_0$, but it is not possible to do it directly when $z_0$ is near a "turn". Intuitively, in order for $z_0$ to be near a "turn", $z_{-i}$ must be near the critical set for some $i > 0$. This motivates the following considerations.

*Case 1.* There exists arbitrarily large $i$ such that $d_{\mathcal{C}}(z_{-i}) < \rho^k$ for $k \approx K_0(\log \|DT\|)\theta i$ where $K_0$ is to be specified shortly. Let $\varepsilon > 0$ be given, and let $i$ and $k$ have the relationship above with $\|DT\|^{-i} < \varepsilon^{10}$. Let $j$ be such that $z_{-i} \in \hat{Q}^{(j)} \setminus Q^{(j)}$. Then $j \geq k$. We wish to apply Lemma 10.3 to $z_0' = z_{-i}$ with $\varepsilon' = \|DT\|^{-i}\varepsilon$ and $H$ bounded by $\partial R_j$. This result transported back to $z_0$ proves the proposition. To satisfy the hypotheses of Lemma 10.3 at $z_0'$, it suffices to check that the Hausdorff distance between the two horizontal boundaries of $\hat{Q}^{(j)}$ is $< (\|DT\|^{-i}\varepsilon)^{10}$. This is true provided $K_0$ is chosen to satisfy the inequality

$$b^{\frac{k}{4}} = (b^{\frac{1}{4}K_0\theta \log \|DT\|})^i = (\|DT\|^{\log b^{\frac{1}{4}K_0\theta}})^i < \|DT\|^{-11i} = (\|DT\|^{-i}\varepsilon)^{10}.$$

*Case 2.* Not Case 1. Note that this means that $z_{-i}$ approaches $\mathcal{C}$ extremely slowly (if at all) as $i \to \infty$. First we observe that with $d_{\mathcal{C}}(z_{-i}) >> b^{\frac{i}{2}}$, $z_0$ is out of all fold periods from the past. To arrange for the scenario of Lemma 10.3 at $z_0$, we will show:

(i) there exist $\kappa = \mathcal{O}(1)$ and arbitrarily large $i$ such that $\|DT^j(z_{-i})\binom{0}{1}\| \geq \kappa^j$ for all $j \leq i$;

(ii) the stable curves near $z_{-i}$ when mapped forwards bring with them to $z_0$ a pair of curves from $\partial R_n$ with $z_0$ sandwiched in between;

(iii) these curves are $C^2(b)$, they have a minumum length $\varepsilon_1$ independent of $i$ and their Hausdorff distance can be made as small as need be by choosing $i$ large.

We prove (i). Leaving the $\inf_i d_{\mathcal{C}}(z_{-i}) > 0$ case as an exercise, we consider $i$ with $d_{\mathcal{C}}(z_{-i}) \leq d_{\mathcal{C}}(z_{-j})$ for all $0 < j \leq i$. Suppose $d_{\mathcal{C}}(z_{-i}) \approx e^{-\mu}$, so that the ensuing bound period is $> K^{-1}\mu$. Let $w_j = DT^j(z_{-i})\binom{0}{1}$, and let $z_{-i+n}$ be the next free return. Then $\|w_j\| \geq 1$ for $j \leq n$. We argue that $w_n$ splits correctly: If $z_{-i+n} \in \mathcal{C}^{(n)}$, then $d_{\mathcal{C}}(z_{-i+n}) \geq d_{\mathcal{C}}(z_{-i}) \approx e^{-\mu} >> b^{\frac{1}{20}K^{-1}\mu} \geq b^{\frac{1}{20}n}$; if $z_{-i+n} \notin \mathcal{C}^{(n)}$, then it is $\in \hat{Q}^{(j)} - Q^{(j)}$ for some $j < n$. In both cases, Lemma 7.1 applies, and we have $\|w_{n+1}^*\| \geq e^{\frac{cn}{3}} e^{-\mu} \geq e^{(\frac{c}{3}-K)n}$. Since the situation at subsequent free returns is clearly

improved ($d_{\mathcal{C}}(\cdot) \geq d_{\mathcal{C}}(z_{-i})$ and the derivative has built up), we have $\|w_j\| \geq e^{(\frac{c}{3}-K)j}$ for all $j \leq i$.

To prove (ii), suppose $z_{-i} \in \hat{Q}^{(k)} \setminus Q^{(k)}$ for some $k$. We consider the stable curve of order $i$ through $z_{-i}$ and let $\zeta_0$ be its intersection with the upper boundary of $\hat{Q}^{(k)}$. A subsegment $\gamma_0$ of this upper boundary centered at $\zeta_0$ is constructed by iterating forward $i$ times and trimming whenever necessary so that $T^j\gamma_0$ stays inside three consecutive $I_{\mu\ell}$ for all $j \leq i$. Clearly, stable curves of order $i$ can be constructed through all points in $\gamma_0$, and these curves "tie together" the two subsegments of $\partial\hat{Q}^{(k)}$.

We leave it as an exercise to show the existence of $\varepsilon_1$ (which depends only on the slow rate of approach to $\mathcal{C}$ in backward time). The curves brought in are sebsegments of $\partial R_{k+i}$ and they are out of all fold periods. This completes the proof of Proposition 10.1′. □

## 10.4 Proof of Theorem 2(1)(iii)

We explain how $\Omega = \overline{\cup_{\varepsilon>0}\Omega_\varepsilon}$ follows readily from the ideas in the last two subsections and the surjectivity condition (*) in Sect. 1.2.

In view of Proposition 10.1′, it suffices to show that every $S \in \mathcal{T}$ contains a point in $\Omega_\varepsilon$ for some $\varepsilon > 0$. Recall the way monotone branches in $\mathcal{T}$ are constructed. Given $S \in \mathcal{T}$, let $\ell > 0$ be the smallest integer such that $T^{-\ell}S \notin \mathcal{T}$. Then $T^{-\ell}S$ contains half of some $Q^{(k)}$. Let $H$ be the middle half of $T^{-\ell}S \cap Q^{(k)}$, with length $\frac{1}{4}\rho^k$. An argument similar to that in Lemma 10.3 but carried on indefinitely in time gives a sequence of domains $H \supset H'_1 \supset H'_2 \supset \cdots$ and a curve $\omega_0 \subset \cap_{n\geq 1}H'_n$ with the following properties:

— $\omega_0$ connects the top and bottom boundaries of $Q^{(k)} \cap T^{-\ell}S$;

— there exists $\varepsilon > 0$ such that $\forall z \in \omega_0,\ d_{\mathcal{C}}(z_n) \geq \varepsilon\ \forall n \geq 0$.

To finish, it suffices to produce $\hat{z}_0 \in \omega_0$ such that $\hat{z}_{-i} \notin \mathcal{C}^{(0)}\ \forall i > 0$. Let $D_i$ be the component of $R_0 \setminus \mathcal{C}^{(0)}$ between the $i$-th and $(i+1)$-st components of $\mathcal{C}^{(0)}$, and let $\hat{D}_i$ be the union of $D_i$ with the two components of $\mathcal{C}^{(0)}$ adjacent to it. Then we may assume from condition (*) that for every $i$, there exists $j$ such that $T(D_j) \cap \hat{D}_i$ contains a horizontal strip traversing the full length of $\hat{D}_i$. Suppose $\omega_0 \subset \hat{D}_i$, and let $j$ be as above. Then there is a subsegment $\omega_1 \subset \omega_0$ such that $T^{-1}\omega_1 \subset D_j$ and connects the top and bottom boundaries of $D_j$. Similarly, we produce for $n = 2, 3, \cdots$ segments $\omega_n \subset \omega_{n-1}$ such that $T^{-n}\omega_n$ is contained in some $D_{j(n)}$ and connects the two horizontal boundaries of $D_{j(n)}$. Let $\hat{z}_0 \in \cap_{n\geq 0}\omega_n$. □

## 10.5 Existence of Equilibrium states

This is a corollary to the symbolic dynamics we have developed. Let $\varphi : R_0 \to \mathbb{R}$ be a continuous function, and let $P(T;\varphi)$ denote the **topological pressure** of $T$ for the

potential $\varphi$. (See e.g. [Wa], Chapter 9, for definitions and basic facts.) A well known variational principle says that

$$P(T;\varphi) \;=\; sup \;\; P_\nu(T;\varphi)$$

where the supremum is taken over all $T$-invariant Borel probability measures $\nu$ and

$$P_\nu(T;\varphi) := h_\nu(T) + \int \varphi d\nu,$$

where $h_\nu(T)$ denotes the metric entropy of $T$ with respect to $\nu$. An invariant measure for which this supremum is attained is called an **equilibrium state** for $(T;\varphi)$.

Let $\sigma : \Sigma \to \Sigma$ and $\pi : \Sigma \to \Omega$ be as in Theorem 6.

**Proof of Corollary 2:** Let $\varphi : R_0 \to \mathbb{R}$ be given. We need to prove that there exists $\nu$ such that $P_\nu(T;\varphi) = P(T;\varphi)$. Let $\tilde{\varphi}$ be the function on $\Sigma$ defined by $\tilde{\varphi} = \varphi \circ \pi$. Then $P(T;\varphi) = P(T|\Omega;\varphi|\Omega) \leq P(\sigma;\tilde{\varphi})$. Since $\sigma : \Sigma \to \Sigma$ has a natural finite generator without boundary, $(\sigma,\tilde{\varphi})$ has an equilibrium state which we call $\tilde{\nu}$. Let $\nu = \pi_*\tilde{\nu}$. It suffices to show that $P_\nu(T|\Omega;\varphi|\Omega) = P_{\tilde{\nu}}(\sigma;\tilde{\varphi})$. This follows from the fact that $\pi$ is one-to-one over $\Omega \setminus \cup T^i\mathcal{C}$, and $\mu(\pi^{-1}(\cup T^i\mathcal{C})) = 0$ for any $\sigma$-invariant probability measure $\mu$ because $\sigma^i(\pi^{-1}\mathcal{C}) \cap \pi^{-1}\mathcal{C} = \emptyset$ for all $i \in \mathbb{Z}$. $\square$

Since the **topological entropy** of $T$, written $h_{\text{top}}(T)$, is equal to $P(T;0)$, the discussion above gives immediately

**Corollary 10.1** *(i) $T$ has an invariant measure of maximal entropy.*

*(ii) Let $N_n$ be the number of distinct blocks of symbols of length $n$ that appear in $\Sigma$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log N_n \;=\; h_{\text{top}}(T).$$

## 10.6 Topological entropy

Topological entropy is, in general, defined in terms of open covers of arbitrarily small diameters, $\varepsilon$-separated or spanning sets. None of the standard definitions is easy to compute with. Corollary 10.1 gives a concrete way to think about this invariant for the class of dynamical systems under consideration. Three other characterizations and estimates of geometric interest are discussed here.

Recall the notion of $\tilde{a}^{(k)}$-addresses for $z \in R_k$ (see Sect. 10.2). For $z_0 \in R_0$, we define its (future) $\tilde{a}$-itinerary to be $(a_i)_0^\infty$ if for each $i$, $\tilde{a}^{(i)}(z_i) = a_i$. These itineraries are clearly not unique. Let

$$\tilde{N}_n \;=\; \text{the number of } n\text{-blocks appearing in the } \tilde{a}\text{-itineraries of points in } R_0,$$

overcounting whenever ambiguities arise, that is, if an orbit has $j$ different admissible $\tilde{a}$-itineraries of length $n$, they will be counted as $j$ distinct blocks in $\tilde{N}_n$. Obviously, $N_n \leq \tilde{N}_n$.

**Lemma 10.4**

$$\limsup_{n\to\infty} \frac{1}{n}\log \tilde{N}_n \;\leq\; h_{\rm top}(T).$$

**Proof:** We fix some arbitrarily small $\varepsilon > 0$, and choose $n_0$ so that

$$\frac{1}{n_0}\log N_{n_0} < h_{\rm top}(T) + \varepsilon \quad \text{and} \quad \frac{1}{n_0}\log(2n_0) < \varepsilon.$$

Let $n_1 > n_0$ be large enough that $b^{\frac{n_1}{10}}\|DT\|^{n_0} < e^{-\beta n_0}$, so that no orbit segment in $R_0$ of length $\leq n_0$ can pass through the region $D := \{\xi_0 \in \mathcal{C}^{(n_1)} : |\xi_0 - \hat{z}_0| < b^{\frac{n_1}{10}}$ for some $\hat{z}_0 \in \mathcal{C} \cap Q^{(n_1)}(\xi_0)\}$ more than once. For each $z_0$, let $S_{z_0} = T^{-n_0}S(z_{n_0}, 2n_1)$ where $S(z_{n_0}, 2n_1)$ is as in Lemma 10.2(ii). By part (i) of the same lemma, $S_{z_0}$ is a neighborhood of $z_0$. Let $n_2 > n_1$ be such that $R_{n_2} \subset \cup_{z_0\in\Omega} S_{z_0}$. Define

$$\begin{aligned}\tilde{N}(n_2, n_2+n_0) \;=\; &\text{the number of distinct blocks of } [a_{n_2}, \cdots, a_{n_2+n_0-1}] \\ &\text{that appear in the } \tilde{a}\text{-itineraries of all points in } R_0.\end{aligned}$$

**Claim 10.1** $\tilde{N}(n_2, n_2+n_0) \leq 2n_0 N_{n_0}$.

*Proof of Claim 10.1:* Let $\xi_0 \in R_0$, and let $(a_i)$ be any one of its $\tilde{a}$-itineraries. Let $\xi_{n_2} \in S_{z_0}$ for some $z_0 \in \Omega$, and let $\iota(z_0) = (b_i)$. We compare the two blocks $[a_{n_2}, \cdots, a_{n_2+n_0-1}]$ and $[b_0, \cdots, b_{n_0-1}]$. The $i$th entry of the first block is an $\tilde{a}^{(n_2+i)}$-address of $\xi_{n_2+i}$. Since $\xi_{n_2+n_0} \in S = S(z_{n_0}, 2n_1)$, it follows from Lemma 10.2 that the $i$-th entry of the second block is an $\tilde{a}^{(n(S)-n_0+i)}$-address of $\xi_{n_2+i}$ where $n(S)$ is such that $S \in \mathcal{T}_{n(S)}$. Since the indices in both of these $\tilde{a}$-addresses exceed $n_1$, they may differ only if $\xi_{n_2+i} \in D$. This can happen at most once in the time period in question. In other words, $[a_{n_2}, \cdots, a_{n_2+n_0-1}]$ and $[b_0, \cdots, b_{n_0-1}]$ can differ in at most one entry, and the difference is either $+1$ or $-1$. Since $[b_0, \cdots, b_{n_0-1}]$ is one of the sequences counted in $N_{n_0}$, the claim is proved.

$\diamondsuit$

Similar reasoning shows that $\tilde{N}(n_2 + kn_0, n_2 + (k+1)n_0) \leq 2n_0 N_{n_0}$ for all $k \geq 0$, giving

$$\tilde{N}_{n_2+kn_0} \;\leq\; K^{n_2} \cdot (2n_0 N_{n_0})^k.$$

This combined with the properties we imposed on $n_0$ at the beginning of the proof gives the desired inequality. $\square$

To complete the proof of Theorem 7(i), recall that $P_n$ is the number of fixed points of $T^n$ in $\Omega$.

**Lemma 10.5**

$$\lim_{n\to\infty} \frac{1}{n}\log P_n \;=\; h_{\rm top}(T).$$

**Proof:** Since no point in $\mathcal{C}$ is periodic, there is a one-to-one correspondence between the fixed points of $T^n$ and the periodic symbol sequences of period $n$ in $\Sigma$, proving "$\leq$" in the lemma. That

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n \; > \; h_{\text{top}}(T) - \varepsilon$$

for every $\varepsilon > 0$ follows from a general theorem of Katok for all $C^2$ surface diffeomorphisms [K]. □

Perhaps the most concrete geometric quantity of all is the rate of growth of the number of monotone segments of a curve such as $\partial R_0$. Our next lemma compares this growth rate to the topological entropy of $T$. Let $\partial R_0^+$ and $\partial R_0^-$ denote the two components of $\partial R_0$, and define

$$M_n^{\pm} \; = \; \text{the number of monotone segments in } \partial R_n^{\pm}$$

where "monotone segments" are as defined in Sect. 9.1.

**Proof of Theorem 7(ii):** First we prove $M_n^{\pm} \leq \tilde{N}_n$. This follows from the fact that for every monotone segment $\gamma$ in $\partial R_n^{\pm}$, $\iota(\gamma)$ is counted in $\tilde{N}_n$, and the mapping $\gamma \mapsto \iota(\gamma)$ is injective.

To prove the second inequality, we associate to each $n$-block $[a_{-n}, \cdots, a_{-1}]$ that appears in $\Sigma$ first a point $z_0 \in \Omega$ with $a(z_{-i}) = a_{-i}$ and then a monotone branch $S = S(z_0, n)$ as in Lemma 10.2. Then $S \in \mathcal{T}_k$ for some $k$ with $n \leq k \leq n(1+\varepsilon_0)$, $\varepsilon_0 = 3(\log \frac{1}{b})^{-1}$. We remarked at the end of Sect. 9.3 that every $S \in \mathcal{T}$ has a boundary component $\gamma^+$ in $\partial R_k^+$ and one in $\partial R_k^-$. We have thus defined, for each fixed $n$, a mapping from the set of $n$-blocks in $\Sigma$ to the set of monotone segments of $\partial R_k^+$, $n \leq k \leq n(1+\varepsilon_0)$. This mapping is clearly injective since $\iota(\gamma^+) = \iota(S) = [*, \cdots, *, a_{-n}, \cdots, a_{-1}]$, proving

$$N_n \; \leq \sum_{n \leq k \leq n(1+\varepsilon_0)} M_k^+.$$

From this one deduces easily that

$$\lim \frac{1}{n} \log N_n \; \leq \; (1+\varepsilon_0) \; \liminf \frac{1}{(1+\varepsilon_0)n} \log M^+_{(1+\varepsilon_0)n}.$$

□

# Appendix A Examples

## A.1 Attractors arising from interval maps including the Hénon attractors

**Reduction of Theorem 8 to Theorems 1–7:** Let $I_0$ be a closed interval such that $f(I) \subset \text{int}(I_0) \subset I_0 \subset \text{int }(I)$, and let $J_1$ and $J_2$ be the two components of $I \setminus I_0$. Choosing $b_0 << |J_1|, |J_2|$, one obtains easily from the formulas for $T_{a,b}$ in Sect. 1.1 that there exist $K > 0$ and $\hat{\Delta} := [a_0, a_1] \times (0, b_0]$ such that for all $(a, b) \in \hat{\Delta}$, $T_{a,b}$ maps $R := I \times [-Kb, Kb]$ strictly into $I_0 \times [-Kb, Kb]$.

Our plan is to replace $\partial I \times [-Kb, Kb]$ by two curves $\omega_1$ and $\omega_2$ so that each $\omega_i \subset J_i \times [-Kb, Kb]$, joins the top and bottom boundaries of $R$, and lies on the stable curve of a periodic orbit. We may assume that these periodic orbits stay outside of $\mathcal{C}^{(0)}$. Replacing $R$ by $R_0$, the subregion of $R$ bounded by $\omega_1$ and $\omega_2$, the situation is now virtually indistinguishable from that of the annlus maps treated in Theorems 1–7: the top and bottom boundaries of $R_0$ play the role of $\partial R_0$ in the previous situation, and the left and right boundaries shrink exponentially as we iterate. (There are small differences, such as the existence of monotone branches with one end bounded by images of $\omega_i$. These differences are inessential.)

To produce $\omega_1$ and $\omega_2$, we claim that pre-periodic points of $f$ are dense in $I$. This claim is justified as follows. First, Misiurewicz maps have no homtervals, so that there is a coding of the orbits of $f$ by a subshift $\sigma : \Sigma \to \Sigma$ with the property that each element of $\Sigma$ corresponds to the itinerary of exactly one point in $I$. Second, $\Sigma$ is the closure of $\cup_n \Sigma_n$ where $\{\Sigma_n\}$ is an increasing sequence of subshifts of finite type, and third, pre-periodic points are dense in shifts of finite type.

To finish, we fix pre-periodic points $p_1$ and $p_2$ of $f$ near the middle of $J_1$ and $J_2$. Shrinking $\hat{\Delta}$ if necessary, we may assume that for $T_{a,b}$ with $(a, b) \in \hat{\Delta}$, the periodic orbits related to $p_1$ and $p_2$ persist and the stable curves through the continuation of $p_i$ have the desired properties. This is possible because the slopes of these stable curves are bounded away from zero (see Lemma 2.9(a)). □

**Proof of Corollary 3:** For the quadratic family, the transversality condition in Step II in Sect. 1.1 hold at all Misiurewicz points [T]. The nondegeneracy condition in Step IV is obviously satisfied. (To ensure that $f(I) \subset \text{int}(I)$ for some $I$ in the case $a^* = 2$, consider $a$ slightly less than 2.) □

## A.2 Homoclinic bifurcations

We verify here the conditions in Sect. 1.1 and condition (**) in Sect. 1.2 for homoclinic bifurcations in 2-dimensions, setting the stage to apply Theorems 1–7. See Sect. 1.5 for a more detailed description of the bifurcation in question.

Following [PT], pages 47-51, we assume that linearizing coordinates have been chosen in which $g_\mu,\ \mu \in [0, \mu^*]$, has the following properties:

(i) On $\{|\xi|, |\eta| < 2\}$, $g_\mu$ is the linear map

$$g_\mu(\xi, \eta) = (\sigma_\mu \xi, \lambda_\mu \eta)$$

where $0 < \lambda_\mu < 1 < \sigma_\mu$, $\lambda_\mu \sigma_\mu < 1$, and $\lambda_\mu, \sigma_\mu$ depend continuously on $\mu$.

(ii) There exists $N \in \mathbb{Z}^+$ such that $g_0^N$ maps the point $(1, 0)$ to $(0, 1)$, carrying the unstable curve at $(1, 0)$ to a curve making a quadratic tangency with the stable curve at $(0, 1)$. Near $(1, 0)$, $g_\mu^N$ has the form

$$g_\mu^N(\xi, \eta) = (\alpha(\xi - 1)^2 + \beta\eta + \gamma\mu + H_1(\mu, \xi, \eta),\ 1 + H_2(\mu, \xi, \eta)) \tag{16}$$

where $\alpha, \beta, \gamma \neq 0$ are constants. Furthermore, we have that at $(\mu, \xi, \eta) = (0, 1, 0)$, $H_1 = H_2 = 0$, $\partial_\xi H_1 = \partial_\eta H_1 = \partial_\mu H_1 = 0$ and $\partial_{\xi\xi} H_1 = \partial_{\xi\mu} H_1 = \partial_{\mu\mu} H_1 = 0$.

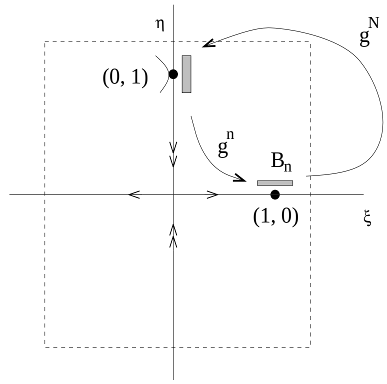



Figure 9  Attractors arising from homoclinic bifurcations

It is not hard to see that for each fixed $n$, $n$ large, there exist a box $B_n$ (with $\text{diam}(B_n) \to 0$ as $n \to \infty$) and a range of parameters $\mu$ (also depending on $n$) such that $(g_\mu^n \circ g_\mu^N)(B_n) \subset B_n$. The attractors of interest to us have $(n + N)$ components permuted cyclically by $g_\mu$, with one of these components residing in $B_n$.

To maneuver $g^n \circ g^N$ into the setting in Sect. 1.1, we apply the coordinate transformation $\Phi = \Phi_2 \circ \Phi_1$ where

$$\Phi_1(\xi, \eta) = (\xi - 1, \eta - \lambda^n), \qquad \Phi_2(\xi, \eta) = (-\frac{\sigma^n}{a}\xi, -\frac{\sigma^{2n}}{a}\eta).$$

The purpose of $\Phi_1$ is to shift the center of $B_n$ to the origin. The map $\Phi_2$ magnifies the attractor to unit length; its scaling in the $\eta$-direction is chosen with the standard quadratic family in mind. A straightforward computation yields

$$T := \quad \Phi \circ g^n \circ g^N \circ \Phi^{-1} : \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{a}[\sigma^n - \sigma^{2n}(\lambda^n + \mu)] - ax^2 + y - \frac{\sigma^{2n}}{a} H_1(\mu, \Phi^{-1}(x, y)) \\ -\frac{\sigma^{2n}}{a} \lambda^n H_2(\mu, \Phi^{-1}(x, y)) \end{pmatrix}.$$

Letting $a = \Psi(\mu) := \sigma^n - \sigma^{2n}(\lambda^n + \mu)$ and $\tilde{H}_i(a, x, y) := H_i(\mu, \Phi^{-1}(x, y)),\ i = 1, 2$, we have

$$T : \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + y - \frac{\sigma^{2n}}{a} \tilde{H}_1(a, x, y) \\ -\frac{\sigma^{2n}}{a} \lambda^n \tilde{H}_2(a, x, y) \end{pmatrix}.$$

Since $\mu = \sigma^{-n} - a\sigma^{-2n} - \lambda^n$, the range of $a$ of interest to us, namely $a \in [1.5, 2)$ (see Appendix A.1), corresponds to a subset of $(0, \mu^*]$ for $n$ large.

What we have so far is a 1-parameter family $\{T_a\}$, which we regard as defined on $U := \{|x|, |y| < 2\}$. The role of $b \to 0$ here is played by $n \to \infty$. Our next task is to choose $b$ (as a function of $n$) in such a way that $T_{a,b}$ has the form

$$T_{a,b} : \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + y + bu \\ bv \end{pmatrix}$$

where $u = u(a, x, y)$ and $v = v(a, x, y)$ have uniformly bounded $C^3$-norms. This will put us in the setting of Theorem 8 (see the proof of Corollary 3).

We begin by examing the $C^3$-norms of $\sigma^{2n}\tilde{H}_1$ and $\sigma^{2n}\lambda^n\tilde{H}_2$. Using the facts that the leading terms in $H_1$ are $\eta(\xi - 1 + \eta + \mu)$, and that $|\xi| < 3\sigma^{-n}$ and $|\eta| < 3\sigma^{-2n}$ for $(\xi, \eta) \in \Phi^{-1}(U)$, we have $\|\tilde{H}_1\|_{C^0} = \mathcal{O}(\sigma^{-3n})$. Similarly, $\|\tilde{H}_2\|_{C^0} = \mathcal{O}(\sigma^{-n})$. Let $\partial^i, i = 1, 2, 3$, denote any one of the $i$-th partial derivatives. Using again the special form of $H_1$ and the nature of the coordinate transformations $\Phi$ and $\Psi$, we have $\|\partial^i\tilde{H}_1\| = \mathcal{O}(\sigma^{-3n})$ and $\|\partial^i\tilde{H}_2\| = \mathcal{O}(\sigma^{-n})$. Together this gives

$$\|\sigma^{2n}\tilde{H}_1\|_{C^3} < K\sigma^{-n}, \quad \|\sigma^{2n}\lambda^n\tilde{H}_2\|_{C^3} < K(\sigma\lambda)^n.$$

The following choices of $b$ therefore give the desired result:

If $\sigma^2\lambda \leq 1$, let $b = \sigma^{-n}$.
If $\sigma^2\lambda \geq 1$, let $b = (\sigma\lambda)^n$.

This completes the verification of the conditions in Sect. 1.1 for the family $\{T_{a,b}\}$. We finish with the observation that all the results in Section 1 that assume (**) are valid in the present setting: In the case $\sigma^2\lambda \leq 1$, $|\det(DT)| \sim b$, so (**) is satisfied. When $\sigma^2\lambda \geq 1$, $|\det(DT)| \sim (\sigma\lambda)^n = b^\eta$ where $\sigma^{-1} = (\sigma\lambda)^\eta$. This is condition $(**)'$, a variant of (**) discussed in Sect. 7.2

# Appendix B Computational Proofs

## B.1 Linear algebra (Sect. 2.1)

**Sublemma B.1** *Let $e$ be a unit vector in the most contracted direction of*

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

*with $\|Me\| = \lambda^{min}$. Then*

$$\begin{aligned} e &= \pm\frac{1}{\rho}(C^2 + D^2 - (\lambda^{min})^2,\ -(AC + BD))\ , &(17)\\ Me &= \pm\frac{1}{\rho}(-A(\lambda^{min})^2 + D\det(M),\ -B(\lambda^{min})^2 - C\det(M)) &(18)\end{aligned}$$

*and*

$$(\lambda^{min})^2 = \frac{1}{2}(A^2 + B^2 + C^2 + D^2 - \sqrt{(A^2 + B^2 + C^2 + D^2)^2 - 4(\det(M))^2})$$

*where $\rho$ is the normalizing constant in (17).*

The proof is left as an easy exercise.

**Proof of Lemma 2.1:** Let $O_1$ and $O_2$ be orthogonal matrices such that

$$O_2 M^{(i-1)} O_1 = \begin{pmatrix} \lambda_{i-1}^{min} & 0 \\ 0 & \lambda_{i-1}^{max} \end{pmatrix}.$$

Then the tangent of the angle between $e_{i-1}$ and $e_i$ is given by the slope of the most contracted direction of the matrix

$$M_i O_2^{-1} \begin{pmatrix} \lambda_{i-1}^{min} & 0 \\ 0 & \lambda_{i-1}^{max} \end{pmatrix} := \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} \lambda_{i-1}^{min} & 0 \\ 0 & \lambda_{i-1}^{max} \end{pmatrix} = \begin{pmatrix} \lambda_{i-1}^{min}A & \lambda_{i-1}^{max}C \\ \lambda_{i-1}^{min}B & \lambda_{i-1}^{max}D \end{pmatrix}.$$

From Sublemma B.1, we see that the slope in question is equal to

$$\frac{(AC+BD)\lambda_{i-1}^{min}\lambda_{i-1}^{max}}{(C^2+D^2)(\lambda_{i-1}^{max})^2-(\lambda_i^{min})^2}\ .$$

This is $\leq \left(\frac{Kb}{\kappa^2}\right)^{i-1}$ because $\lambda_{i-1}^{min}\lambda_{i-1}^{max} = |\det(M^{(i-1)})| < b^{i-1}$, $\lambda_i^{min} < (\frac{b}{\kappa})^i$ and $(C^2+D^2)(\lambda_{i-1}^{max})^2 > K^{-1}\kappa^{2(i-1)}$, the last inequality being a consequence of the fact that $\|M^{(i)}\| > \kappa^i$ and $(A^2+B^2)(\lambda_{i-1}^{min})^2 < K(\frac{b}{\kappa})^{2(i-1)}$. □

Before giving the proof of Corollary 2.2 we state another lemma the proof of which is also a straightforward computation.

**Sublemma B.2** *Let*

$$M_i = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \qquad M^{(j)} = \begin{pmatrix} A_j & C_j \\ B_j & D_j \end{pmatrix}, \qquad j = i-1, i.$$

*Then*

$$\begin{aligned} \|e_i \times e_{i-1}\| &= \frac{1}{\rho^{(i)}\rho^{(i-1)}}\,|\det(M^{(i-1)})[(AC+BD)(C_{i-1}^2+D_{i-1}^2) \\ &\quad +(A^2+B^2-C^2-D^2)C_{i-1}D_{i-1}] + \Delta_i\,| \end{aligned} \tag{19}$$

*where $\rho^{(i-1)}$ and $\rho^{(i)}$ are the normalizing constants for $e_{i-1}$ and $e_i$ as in Sublemma B.1, and*

$$\Delta_i = -(\lambda_i^{min})^2(A_{i-1}C_{i-1}+B_{i-1}D_{i-1}) + (\lambda_{i-1}^{min})^2(A_iC_i+B_iD_i).$$

Observe that each the terms in the numerator of (19) has a factor $|\det(M^{(i-1)})|$, $\lambda_{i-1}^{min}$ or $\lambda_i^{min}$, all of which are $\leq (\frac{b}{\kappa})^{i-1}$. Observe also that if both $e_{i-1}$ and $e_i$ are nearly parallel to the $x$-axis, then $\rho^{(i)}, \rho^{(i-1)}$ are $> K^{-1}\kappa^{2i}$ (see the proof of Lemma 2.1).

**Proof of Corollary 2.2:** We begin with some useful derivative estimates. First, we claim that

$$\|\partial^1 M^{(i)}\| < K^i. \tag{20}$$

This is because $\partial^1 M^{(i)}$ is the sum of $i$ terms of the form $M_i \cdots M_{j+1}(\partial^1 M_j)M_{j-1}\cdots M_1$ and the norm of this product is $< K_0^{2i}$. A similar argument gives

$$|\partial^1 \det M^{(i)}| \leq (Kb)^i. \tag{21}$$

Since $\lambda_i^{max} = \|M^{(i)}\|$, it follows from (20) that $|\partial^1 \lambda_i^{max}| < K^i$; and since $\lambda_i^{min} = |\det M^{(i)}|/\lambda_i^{max}$, we have $|\partial^1\lambda_i^{min}| < (\frac{Kb}{\kappa^2})^i$.

Pre-composing with a suitable orthogonal matrix as in the proof of Lemma 2.1, we may assume that $\rho^{(i)}, \rho^{(i-1)}$ are $> K^{-1}\kappa^{2i}$. The estimate for $\partial^j\theta_1$ is obtained by differentiating (17). To prove (4), we differentiate (19), and observe using the inequalities above that after differentiation, the numerator is the sum of a finite number of terms each one of which is bounded above by $(\frac{Kb}{\kappa^2})^{i-1}$.

To prove (5), we write

$$\begin{aligned} M^{(i)}e_n &= M^{(i)}e_i + M^{(i)}(e_n - e_i) \\ &= M^{(i)}e_i + \sum_{k=i}^{n-1} M^{(i)}(e_{k+1}-e_k) \end{aligned}$$

and take partial derivative one term at a time. First we have

$$\partial^1 M^{(i)}(e_{k+1}-e_k) = \partial^1 M^{(i)} \cdot (e_{k+1}-e_k) + M^{(i)} \cdot \partial^1(e_{k+1}-e_k).$$

The norm of the first term on the right side is bounded by $(\frac{Kb}{\kappa^2})^k$ because $\|\partial^1 M^{(i)}\| \leq K^i$ and $\|e_{k+1}-e_k\| < (\frac{Kb}{\kappa^2})^k$. The norm of the second term is bounded by $(\frac{Kb}{\kappa^2})^k$ according to (4). It remains to show $\|\partial^1 M^{(i)} e_i\| < (\frac{Kb}{\kappa^2})^i$. This follows by differentiating (18) and using the inequalities above. The proof for $j=2$ is similar. □

**Sublemma B.3** *Let $M_i$ and $M_i'$ be as in Lemma 2.2, let $m < \frac{n}{2}$, and write*

$$M_{i,m} = M_{i+m} M_{i-1+m} \cdots M_m \ , \quad M_{i,m}' = M_{i+m}' M_{i-1+m}' \cdots M_m'.$$

*Then*

$$\|M_{i,m} - M_{i,m}'\| < \frac{1}{4}(K\lambda)^m \tag{22}$$

*for all $i$, $0 \leq i \leq m$.*

**Proof:** Set $\rho_k = \|M_{k,m} - M_{k,m}'\|$. Then

$$\begin{aligned} M_{k+1,m} - M_{k+1,m}' &= M_{k+1+m} M_{k,m} - M_{k+1+m}' M_{k,m}' \\ &= M_{k+1+m}(M_{k,m} - M_{k,m}') + (M_{k+1+m} - M_{k+1+m}') M_{k,m}'. \end{aligned}$$

Since $\|M_{k,m}'\| < K_0^k$ and $\|M_{k+1+m} - M_{k+1+m}'\| < \lambda^{k+m}$, we have

$$\rho_{k+1} \leq K\rho_k + K^k \lambda^{m+k},$$

which implies (22). □

**Proof of Lemma 2.2:** ([BC2], p. 108): We prove the assertion for all the indices that are powers of two and leave the rest as an exercise. To prove (b), write $m_j = 2^j$, and let

$$u_j = \frac{w_{m_j}}{\|w_{m_j}\|}, \quad u_j' = \frac{w_{m_j}'}{\|w_{m_j}'\|}$$

where $w_{m_j} = M^{(m_i)} w$ and $w_{m_j}' = M'^{(m_i)} w$. We will show inductively that

$$\|u_j \times u_j'\| < \lambda^{\frac{m_j}{4}}. \tag{23}$$

Assume that (23) is true up to index $j$. Let

$$A = M_{m_{j+1}-m_j, m_j} \quad \textit{and} \quad A' = M_{m_{j+1}-m_j, m_j}'.$$

Since $\|w_{m_j}\| < K^{m_j}$ and $\|w_{m_{j+1}}\| > \kappa^{m_{j+1}}$, we have

$$\|Au_j\| = \frac{\|w_{m_{j+1}}\|}{\|w_{m_j}\|} > \left(\frac{\kappa^2}{K}\right)^{m_j}, \tag{24}$$

$$\|Au_j'\| \geq \|Au_j\| - \|A\|\|u_j - u_j'\| \geq \left(\frac{\kappa^2}{K}\right)^{m_j} - K^{m_j}\lambda^{\frac{m_j}{4}} \geq \frac{3}{4}\left(\frac{\kappa^2}{K}\right)^{m_j}. \tag{25}$$

Writing $\|A'u'_j\| = \|A'u'_j - A'\hat{u}_j + A'\hat{u}_j - A\hat{u}_j + A\hat{u}_j\|$ where $\hat{u}_j = u_j$ if the angle between $u_j$ and $u'_j$ is smaller than $\frac{\pi}{2}$, $\hat{u}_j = -u_j$ otherwise, we obtain $\|A'u'_j\| \geq \|Au_j\| - \|A\|\|u_j \times u'_j\| - \|A - A'\|$. Using Sublemma B.3 to bound $\|A - A'\|$, we again have

$$\| A'u'_j \| \geq \frac{3}{4}\left(\frac{\kappa^2}{K}\right)^{m_j}. \tag{26}$$

We are now ready to prove (23) for index $j+1$:

$$\begin{aligned} \|u_{j+1} \times u'_{j+1}\| &= \frac{\|Au_j \times A'u'_j\|}{\|Au_j\| \cdot \|A'_j u'_j\|} = \frac{\|Au_j \times (A - A + A')u'_j\|}{\|Au_j\| \cdot \|A'_j u'_j\|} \\ &\leq \frac{\|Au_j \times Au'_j\|}{\|Au_j\| \cdot \|A'_j u'_j\|} + \frac{\|Au_j \times (A - A')u'_j\|}{\|Au_j\| \cdot \|A'_j u'_j\|}. \end{aligned}$$

The first term is fine since $\|Au_j \times Au'_j\| = |\det(A)|\ \|u_j \times u'_j\|$ and $|\det(A)| < b^{m_j}$. To estimate the second term, we use Sublemma B.3 and (24)-(26).

To prove (a), we again let $i = 2^k$. Then for $0 < j \leq k$, we have

$$\|w'_{m_{j+1}}\| = \|w'_{m_j}\|\|A'u'_j\| = \|w'_{m_j}\|\|A'u'_j - Au'_j + Au'_j - A\hat{u}_j + A\hat{u}_j\|,$$

so that

$$\begin{aligned} \frac{\|w'_{m_{j+1}}\|}{\|w'_{m_j}\|} &\geq \|Au_j\| - \|A' - A\|\|u'_j\| - \|A\|\|u'_j - \hat{u}_j\| \\ &= \frac{\|w_{m_{j+1}}\|}{\|w_{m_j}\|}\left(1 - \frac{\|w_{m_j}\|}{\|w_{m_{j+1}}\|}(\|A' - A\| + \|A\|\|u'_j - \hat{u}_j\|)\right). \end{aligned}$$

Using Sublemma B.3 to bound $\|A - A'\|$ and part (b) of this lemma to bound $\|\hat{u}_j - u'_j\|$, we obtain

$$\frac{\|w'_{m_{j+1}}\|}{\|w'_{m_j}\|} \geq \frac{\|w_{m_{j+1}}\|}{\|w_{m_j}\|}(1 - 4^{-m_j}),$$

which implies (a). $\square$

## B.2 Stable curves (Sect. 2.2)

On a ball of radius $\frac{\lambda}{2K_0}$ centered at $z_0$, we have $\|DT\| \geq \frac{\kappa}{2}$ so that $e_1$, the field of most contracted directions of $DT$, is well defined. Let $\gamma_1$ be the integral curve to $e_1$ of length $\sim \lambda$ passing through $z_0$.

To construct $\gamma_2$, let $B_1$ be the $\frac{\lambda^2}{2K_0}$-neighborhood of $\gamma_1$. For $\xi \in B_1$, let $\xi'$ be a point in $\gamma_1$ with $|\xi - \xi'| < \frac{\lambda^2}{2K_0}$. Then $|T\xi - Tz_0| \leq |T\xi - T\xi'| + |T\xi' - Tz_0| \leq \frac{\lambda^2}{2} + \frac{Kb}{\kappa^2}\lambda < \lambda^2$, so by Lemma 2.2, $\|DT^2\xi\| \geq \frac{\kappa^2}{2}$. This ensures that $e_2$, the field of most contracted directions for $DT^2$, is defined on all of $B_1$. Let $\gamma_2$ be the integral curve through $z_0$ in $B_1$. We leave it as an exercise to show that the Hausdorff distance between $\gamma_1$ and $\gamma_2$ is $\mathcal{O}(\frac{b}{\kappa^2}\lambda) << \lambda^2$, so that $\gamma_2$ has essentially the same length as $\gamma_1$. This uses the fact that $e_1$ has Lipschitz constant $K$ (Corollary 2.2) and that the angle between $e_1$ and $e_2$ is $< \frac{Kb}{\kappa^2}$ (Corollary 2.1).

Next we let $B_2$ be the $\frac{\lambda^3}{2K_0}$-neighborhood of $\gamma_2$ and repeat the argument above to get $e_3$ and $\gamma_3$. Using the Lipschitzness of $e_2$ and the fact that $\| e_3 \times e_2 \| \leq (\frac{Kb}{\kappa^2})^2$, we conclude again that $\gamma_3$ has essentially the same length as $\gamma_2$. This process is continued for $n$ steps.

## B.3 Curvature estimates (Sect. 2.3)

Recall that

$$k_i(s) = \frac{\|\gamma_i'(s) \times \gamma_i''(s)\|}{\|\gamma_i'(s)\|^3}.$$

Write

$$DT = DT(\gamma_i(s)) = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

and

$$X = \begin{pmatrix} <\nabla A, \gamma_{i-1}'> & <\nabla C, \gamma_{i-1}'> \\ <\nabla B, \gamma_{i-1}'> & <\nabla D, \gamma_{i-1}'> \end{pmatrix}$$

where $< \, , \, >$ is the usual inner product. Since $\gamma_i' = DT \cdot \gamma_{i-1}'$ and $\gamma_i'' = DT \cdot \gamma_{i-1}'' + X \cdot \gamma_{i-1}'$, we have

$$k_i = \frac{1}{\|\gamma_i'\|^3} \|DT \cdot \gamma_{i-1}' \times (DT \cdot \gamma_{i-1}'') + X \cdot \gamma_{i-1}')\| \leq \frac{1}{\|\gamma_i'\|^3}(I + II) \tag{27}$$

where

$$I = |\det(DT)| \cdot \|\gamma_{i-1}' \times \gamma_{i-1}''\|, \quad II = \|DT \cdot \gamma_{i-1}' \times X \cdot \gamma_{i-1}'\|.$$

Term II is degree three homogeneous in $\gamma_{i-1}'$. Moreover, the second component of each vector involved in the cross product has a factor $b$. Thus there exist $K > 0$ such that

$$k_i \leq (b \cdot k_{i-1} + K \cdot b) \cdot \frac{\|\gamma_{i-1}'\|^3}{\|\gamma_i'\|^3}. \tag{28}$$

Lemma 2.4 follows by recursively applying inequality (28).

## B.4 One-dimensional dynamics (Sect. 2.4)

Let $\delta_0 :=$inf$\{d(f^n\hat{x}, C) : \hat{x} \in C, n > 0\}$. We begin with three easy observations:

(i) There exists $k_0 > 0$ such that for all $\delta < \frac{1}{2}\delta_0$, if $x$ is such that $f^n x \in C_\delta$, then $|(f^n)'x| \geq k_0$. This is true because there is an interval $(x_1, x_2)$ containing $x$ on which $f^n$ is monotone and $f^n(x_1, x_2) \supset (\hat{x} - 2\delta, \hat{x} + 2\delta)$ for some $\hat{x} \in C$. It then follows from the negative Schwarzian property that restricted to $f^{-n}(\hat{x} - \delta, \hat{x} + \delta) \cap (x_1, x_2)$, $|(f^n)'| \geq$ some $k_0 > 0$ independent of $x$.

(ii) There exists $\lambda_0 > 1$ such that for all sufficiently small $\delta$, if $d(x, C) < \delta$, then there exists $p = p(x)$ such that $f^i x \notin C_\delta$ for all $i < p$ and $|(f^p)'x| \geq \lambda_0^p$. This is an easy computation using the fact that the forward critical orbits of $f$ are contained in a uniformly expanding invariant set. Let $\hat{p}(\delta) =$inf$\{p(x) : d(x, C) < \delta\}$.

(iii) For all sufficiently small $\delta$, there exist $N_1(\delta) \in \mathbb{Z}$ and $\lambda_1(\delta) > 1$ such that if $x, \cdots, f^n x \notin C_\delta$ for some $n > N_1$, then $|(f^n)'x| \geq \lambda_1^n$. This is proved in [M1].

We now prove the assertion in Lemma 2.5. Fix $\delta_1$ sufficiently small for (i)–(iii) above, and with the property that $\lambda_0^{\hat{p}(\delta_1)} >> k_0^{-1}$. Consider $\delta < \delta_1$ and an orbit segment $x, \cdots, f^n x$ with $f^i x \notin C_\delta$ for $i < n$ and $f^n x \in C_\delta$. To estimate $(f^n)'x$, we let $n_j$ be the $j$th time $f^i x \in C_{\delta_1}$, and let $p_j = p(f^{n_j}x)$. Then $|(f^{p_j})'(f^{n_j}x)| \geq \lambda_0^{p_j}$, and between the times $n_j + p_j$ and $n_{j+1}$, the derivative is bounded below by $\lambda_1(\delta_1)^{n_{j+1}-(n_j+p_j)}$ if $n_{j+1} - (n_j + p_j) > N_1(\delta_1)$, by $k_0$ otherwise. The same estimate holds for the initial stretch up to time $n_1$. Noting that the factor $k_0$ can be absorbed into $\lambda_0^{p_j}$, we see that $|(f^n)'x| \geq e^{\hat{c}_1 n}$ where $e^{\hat{c}_1}$ can be taken to be slightly smaller than $(\min(\lambda_0, \lambda_1(\delta_1))^{\frac{\hat{p}(\delta_1)}{\hat{p}(\delta_1)+N_1(\delta_1)}}$. Also, $\hat{c}_0$ can be taken to be $k_0\lambda^{-N_1(\delta_1)}$. This completes the proof of part (ii) of Lemma 2.5.

To prove (i), let $n_q < n$ be the last time $f^i x \in C_{\delta_1}$, and observe that $|(f^{n-n_q})'(f^{n_q}x)| \geq K^{-1}k_0\delta$ if $n - n_q < N_1(\delta_1)$, $\geq K^{-1}\delta\lambda_1(\delta_1)^{n-n_q}$ otherwise. $\square$

## B.5 Critical points inside $\mathcal{C}^{(0)}$ (Sect. 2.6)

**Proof of Lemma 2.9:** Write

$$\frac{dq_1(s)}{ds} = \partial_x q_1(x,y)\frac{dx(s)}{ds} + \partial_y q_1(x,y)\frac{dy(s)}{ds}. \tag{29}$$

Since $\gamma$ is $b$-horizontal, we have $\frac{dx(s)}{ds} \approx 1$ and $\mid \frac{dy(s)}{ds} \mid < \mathcal{O}(b) \cdot \mid \frac{dx(s)}{ds} \mid$. By (17)

$$q_1(s) = \frac{AC+BD}{C^2+D^2-(\lambda^{min})^2}, \tag{30}$$

so

$$\begin{aligned} \partial_x q_1(x,y) &= \frac{A_xC + AC_x + \mathcal{O}(b)}{C^2+D^2-(\lambda^{min})^2} - 2\frac{(AC+BD)(CC_x+DD_x+\lambda^{min}\lambda_x^{min})}{(C^2+D^2-(\lambda^{min})^2)^2} \\ &:= I + II \end{aligned}$$

where

$$A = F_x + bu_x, \quad C = F_y + bu_y,$$
$$B = bv_x, \quad D = bv_y.$$

We will show that $|I| \geq K^{-1}$ and $|II| = \mathcal{O}(\delta)$. To estimate $I$, observe that the denominator is $> K^{-1}$, and that for $(x,y) \in \mathcal{C}^{(0)}$, $|AC_x| = \mathcal{O}(\delta)$, while $|A_xC| = |F_{xx}F_y|(1+\mathcal{O}(b)) \geq K^{-1}$ since $|F_y| > K^{-1}$ (non-degeneracy condition). Term II follows from the fact that its denominator is $\geq K^{-1}$, and $AC+BD = \mathcal{O}(\delta)$. □

**Proof of Lemma 2.10:** Using the results in Sect. 2.1 and Lemma 2.9, we have that at $\gamma(s)$ with $|s| < (Kb)^{\frac{m}{2}}$, $e_{3m}$ is defined with $|q_{3m} - q_m| < (Kb)^m$ (Lemma 2.2) and $|\frac{d}{ds}q_m| \geq K^{-1}$ (Corollary 2.2 and Lemma 2.9). Let $\tau(s)$ denote the slope of $\gamma'(s)$, and assume for definiteness that $\frac{d}{ds}q_{3m} > 0$. Then

$$\begin{aligned} q_{3m}((Kb)^{\frac{m}{2}}) - \tau((Kb)^{\frac{m}{2}}) &= (q_{3m}((Kb)^{\frac{m}{2}}) - q_m((Kb)^{\frac{m}{2}})) + (q_m((Kb)^{\frac{m}{2}}) - q_m(0)) \\ &\quad + (q_m(0) - \tau(0)) + (\tau(0) - \tau((Kb)^{\frac{m}{2}})) \\ &\geq -(Kb)^m + K^{-1}(Kb)^{\frac{m}{2}} + 0 - K_1 b (Kb)^{\frac{m}{2}} \\ &\geq \frac{K^{-1}}{2}(Kb)^{\frac{m}{2}}. \end{aligned}$$

Similarly, $q_{3m}(-(Kb)^{\frac{m}{2}}) - \tau(-(Kb)^{\frac{m}{2}}) < 0$, giving a unique critical point of order $3m$ in between. □

**Proof of Lemma 2.11:** Let $\tau(s)$ be the slope of the tangent vector to $\gamma$ at $\gamma(s)$, and let $q_m(s)$ be the slope of $q_m$ at $\gamma(s)$. Let $\hat{\tau}(s)$ and $\hat{q}_m(s)$ denote the corresponding quantities at $\hat{\gamma}(s)$. First we claim that

$$\mid \tau(0) - \hat{\tau}(0) \mid \leq 2\sqrt{\varepsilon}. \tag{31}$$

An easy calculation (which we omit) shows that if this was not the case, then $\gamma$ and $\hat{\gamma}$ would meet at $\gamma(s)$ for some $|s| < \sqrt{\varepsilon}$.

Let $\hat{m}$ be the largest integer $j \leq m$ such that $4K_1\sqrt{\varepsilon} < \|DT\|^{-13j}$. Then by Lemma 2.2, $\|DT^i(\gamma(s))\| > \frac{1}{2}$ for $0 < i < \hat{m}$ and $s \in [-4K_1\sqrt{\varepsilon}, 4K_1\sqrt{\varepsilon}]$. This guarantees that $q_{\hat{m}}$ is defined everywhere on $\gamma$ and on $\hat{\gamma}$. Let $\hat{\sigma}(s) := \hat{q}_{\hat{m}}(s) - \hat{\tau}(s)$. We have

$$\begin{aligned} |\hat{\sigma}(0)| &\leq |\hat{q}_{\hat{m}}(0) - q_{\hat{m}}(0)| + |q_{\hat{m}}(0) - q_m(0)| + |q_m(0) - \tau(0)| + |\tau(0) - \hat{\tau}(0)| \\ &< K\varepsilon + (Kb)^{\hat{m}} + 0 + 2\sqrt{\varepsilon} \; < \; 3\sqrt{\varepsilon}. \end{aligned}$$

To prove the existence of a critical point of order $\hat{m}$ on $\hat{\gamma}$, we will compare the signs of $\hat{\sigma}$ at the two end points of $\hat{\gamma}$. First,

$$\hat{\sigma}(4K_1\sqrt{\varepsilon}) = \hat{q}_{\hat{m}}(0) + \frac{d}{ds}q_{\hat{m}}(s_1)\cdot 4K_1\sqrt{\varepsilon} - \hat{\tau}(0) - \frac{d}{ds}\hat{\tau}(s_2)\cdot 4K_1\sqrt{\varepsilon}$$

for some $s_1, s_2 \in [0, 4K_1\sqrt{\varepsilon}]$. This is

$$= \hat{\sigma}(0) + (\frac{d}{ds}q_{\hat{m}}(s_1) + \mathcal{O}(b))\cdot 4K_1\sqrt{\varepsilon}.$$

Since the second term has absolute value $> (K_1^{-1} - \mathcal{O}(b))\cdot 4K_1\sqrt{\varepsilon} > |\hat{\sigma}(0)|$, it follows that $\hat{\sigma}(4K_1\sqrt{\varepsilon})$ has the same sign as $\frac{d}{ds}q_1$. An analogous computation shows that $\hat{\sigma}(-4K_1\sqrt{\varepsilon})$ has the opposite sign as $\frac{d}{ds}q_1$. $\square$

## B.6 Growth of $w_i$ and $w_i^*$ (Sect. 4.2)

**Sublemma B.4** *Let $z_0$ be h-related to $\hat{z}_0 \in \Gamma_{\theta N}$ with bound period $p < \frac{2}{3}N$, and let $w_0 = \binom{0}{1}$. Then for $i \leq p$, $\|w_i^*\| > K^{-1}e^{c''i}$ for some $c'' \approx c$.*

**Proof:** Let $\hat{w}_i^*$ be as defined in (IA6). Then (IA4) and (IA6) together imply that $\|\hat{w}_i^*\| > \frac{c_0}{2}e^{ci}$. The only difference between $\hat{w}_i^*$ and $w_i^*$ is that contractive fields of order $\ell(\hat{z}_i)$ are used for splitting for the former and $\ell(z_i)$ the latter at returns to $\mathcal{C}^{(0)}$. By Lemma 4.2, $\ell(z_i) = \ell(\hat{z}_i) \pm 1$, so that recombination times may differ by one. This is clearly of no consequence. Assuming these times are synchronized, we observe next that $w_i^*$ has the same direction as $\hat{w}_i^*$. This can be seen inductively (using the nested property of fold periods). Finally, a vector split using a field of order $\ell$ or $\ell+1$ may differ in length by a factor of $1 \pm \mathcal{O}(b^\ell)$. Thus $\|w_i^*\| \geq (1-\mathcal{O}(b))^i\|\hat{w}_i^*\|$. $\square$

**Proof of Lemma 4.6:** We may assume $z_i$ is in a fold period, otherwise there is nothing to prove. Let $i_1 < i \leq i_2$ be the longest fold period containing $i$. By Lemma 4.4, which applies also to controlled orbits satisfying $d_{\mathcal{C}}(z_j) > e^{-\alpha j}$, we have $i_2 - i_1 \leq \varepsilon i$. Let $w_{i_1} = Ae + B\binom{0}{1}$ be the usual splitting. Then

$$\|w_i^*\| \leq K^{i-i_1}|B| \leq K^{i-i_1}\|w_{i_2}^*\| = K^{i-i_1}\|w_{i_2}\| \leq K^{i-i_1}(K^{i_2-i}\|w_i\|) \leq K^{\varepsilon i}\|w_i\|.$$

The first "$\leq$" uses the fact that $\frac{\|w_{j+1}^*\|}{\|w_j^*\|} \leq$ some $K$, the second uses Sublemma B.4, and the third $\|DT\| \leq K$. The reverse estimate follows from $\|w_i\| \leq K^{i-i_1}\|w_{i_1}\| \leq K^{i-i_1}d_{\mathcal{C}}(z_{i_1})^{-1}\|w_i^*\|$ and $d_{\mathcal{C}}(z_{i_1}) > e^{-\alpha i}$. $\square$

**Proof of Lemma 4.7:** We give a proof in the case where $j$ exists; the other case is simpler. Let $k \leq i_1 < i_1 + p_1 \leq i_2 < i_2 + p_2 \leq \cdots \leq i_r = j < n$ be defined as follows: we let $i_1$ be the first return to $\mathcal{C}^{(0)}$ at or after time $k$, $p_1$ the bound period of $z_{i_1}$, $i_2$ the first return after $i_1 + p_1$, and so on until $i_r = j$. Writing $k = i_0 + p_0$, we have that $\frac{\|w_n^*\|}{\|w_k^*\|}$ is a product of factors of the following three types:

$$I := \frac{\|w_{i_{s+1}}^*\|}{\|w_{i_s+p_s}^*\|}, \qquad II := \frac{\|w_{i_s+p_s}^*\|}{\|w_{i_s}^*\|} \quad \text{and} \quad III := \frac{\|w_n^*\|}{\|w_j^*\|}.$$

First we prove the lemma assuming that no fold periods initiated before time $k$ expires between times $k$ and $n$. By Lemma 2.8, $I \geq c_0 e^{c_1(i_{s+1}-(i_s+p_s))}$. Since $w_{i_s}^*$ splits correctly, we have, by (IA5), $II \geq K^{-1}e^{\frac{c}{3}p_s}$. Moreover, we may assume that $c_0$ and $K$ above can be absorbed into the

exponential estimate for the bound period $[i_s, i_s + p_s]$. For $III$, let $\ell$ be the fold period initiated at time $j$. If $\ell > n-j$, then $III \geq K^{-1} d_{\mathcal{C}}(z_j) e^{c''(n-j)}$ by Sublemma B.4. If not, we split $w_j^*$ into $w_j^* = Ae_{n-j} + B\binom{0}{1}$, noting that $e_{n-j}$ is defined at $z_j$ by Sublemma B.4 and Lemma 4.6. Then $III \geq K^{-1} d_{\mathcal{C}}(z_j) e^{c''(n-j)} - (Kb)^{n-j}$. The last term is negligible because $d_{\mathcal{C}}(z_j) \sim b^{\frac{\ell}{2}} >> (Kb)^{n-j}$. Altogether, this gives $\frac{\|w_n^*\|}{\|w_k^*\|} \geq K^{-1} d_{\mathcal{C}}(z_j) e^{c'(n-k)}$ for some $c' > 0$ as claimed.

In the rest of the proof, we view contributions from fold periods initiated before time $k$ as perturbations of the estimates above, and verify that they are inconsequential. For $I$, we claim that for each $t$ in question,

$$\frac{\|w_t^*\|}{\|w_{t-1}^*\|} = (1 \pm \mathcal{O}(\sqrt{b}))\frac{\|DT(z_{t-1})w_{t-1}^*\|}{\|w_{t-1}^*\|},$$

so that $I$ has the same estimate as before with possibly a slightly smaller $c_1$. This claim follows from the fact that when a fold period initiated $\ell$ steps earlier expires at time $t$, the vector to rejoin the main term has magnitude $\|DT(z_{t-1})w_{t-1}^*\|\mathcal{O}(b^{\frac{\ell}{2}})$. (See Sect. 2.7)

Next we turn to $III$, which is similar to and a little more complicated than $II$. Given $z_t$ and a vector $u$, we let $u$, $T_*^1(z_t)u$, $T_*^2(z_t)u$, $\cdots$ denote the vectors given by the splitting algorithm for the orbit segment beginning at $z_t$ with initial vector $u$ – neglecting recombinations from fold periods initiated before time $t$. Then

$$w_n^* = T_*^{n-j}(z_j)w_j^* + \sum_{t=j+1}^{n} T_*^{n-t}(z_t)E_t$$

where $E_t$ is the sum of the vectors to be rejoined at time $t$. For fixed $t$, let $\ell$ be the shortest fold period initiated before $k$ to expire at time $t$. From Sect. 2.7, we have $\|E_t\| \leq (Kb)^{\frac{\ell}{2}}\|w_t^*\|$. Also, since this fold period contains the one initiated at $j$, we have, by Sect. 4.1, $K\alpha\ell > (n-j)$. Together this gives

$$\|T_*^{n-t}(z_t)E_t\| \leq K^{n-t}(Kb)^{\frac{\ell}{2}}\|w_t^*\| \leq (Kb^{\frac{1}{K\alpha}})^{n-j}\|w_t^*\|.$$

Assuming inductively that the assertion in the lemma has been proved for shorter time intervals, we have $\|w_t^*\| \leq K d_{\mathcal{C}}(z_{j_t})^{-1}\|w_n^*\|$ where $j_t$ is a return between times $t$ and $n$. Thus

$$\sum_{t=j+1}^{n} \|T_*^{n-t}(z_t)E_t\| < (n-j)(Kb^{\frac{1}{K\alpha}})^{n-j}e^{\alpha(n-j)}\|w_n^*\| << \|w_n^*\|,$$

which together with our earlier estimate on $\|T_*^{n-j}(z_j)w_j^*\|$ gives the disired result. □

**Proof of Lemma 4.8:** The case where $z_k$ is not in a fold period is contained in Lemma 4.7. Let $j$ be the starting point of the largest fold period covering $z_k$. Observe that its length $\ell$ is $< K\theta(n-j) < 2K\theta(n-k)$ because the bound period initiated at time $j$ has expired by time $n$. Then by Lemma 4.7,

$$\|w_n\| > K^{-1}e^{c'(n-j)}\|w_j\| \geq K^{-1}e^{c'(n-j)}K^{-\ell}\|w_k\|.$$

□

**Proof of Lemma 4.5:** The proof proceeds inductively. Consider a bound return $z_i$, and assume that the $w_j^*$-vectors split correctly at *all* returns prior to time $i$. Let $\angle(\cdot,\cdot)$ denote the angle between two vectors, and let $u = \binom{0}{1}$.

**Case 1.** $z_i$ is in a fold period. Let $j < i$ be the largest integer such that the fold period initiated at time $j$ remains in effect at time $i$, and let $\hat{z}_0 = \phi(z_j)$. Then proving $w_i^*$ splits correctly is equivalent to proving

$$\angle(DT^{i-j}(z_j)u,\ \tau(\phi(z_i))\ < \varepsilon_0 d_{\mathcal{C}}(z_i).$$

We compare this inequality to

$$\angle(DT^{i-j}(\hat{z}_0)u,\ \tau(\phi(\hat{z}_{i-j}))\ < \varepsilon_0 d_{\mathcal{C}}(\hat{z}_{i-j}),$$

which we know to be true by (IA3). Suppose $\hat{z}_{i-j} \in \mathcal{C}^{(k)}$. Then

- $\angle(DT^{i-j}(z_j)u,\ DT^{i-j}(\hat{z}_0)u) << e^{-\beta(i-j)} << d_{\mathcal{C}}(z_i)$ by (IA6);
- $\angle(\tau(\phi(z_i),\ \tau(\phi(\hat{z}_{i-j})) < b^{\frac{k-1}{4}} << d_{\mathcal{C}}(z_i)$ by Lemma 4.1;
- $|d_{\mathcal{C}}(z_i) - d_{\mathcal{C}}(\hat{z}_{i-j})| < e^{-\beta(i-j)} + b^{\frac{k-1}{4}} << d_{\mathcal{C}}(z_i)$.

**Case 2.** $z_i$ is not in any fold period. In this case let $j < i$ be the last free return, so that the bound period initiated at $j$ remains in effect at $i$ and $w_i^* = DT^{i-j}(z_j)w_j^*$. We split

$$w_j^*(z_0) = Ae_{i-j} + Bu;$$

$e_{i-j}(z_j)$ is defined (even though $i - j > \ell(z_j)$) by (IA6) and Lemma 4.6. We argue as above that $DT^{i-j}(z_j)u$ splits correctly at $z_i$. It remains to check that adding $A \cdot DT^{i-j}(z_j)e_{i-j}$ will only change the angle of $B \cdot DT^{i-j}(z_j)u$ by $<< e^{-\alpha(i-j)} < d_{\mathcal{C}}(z_i)$. This is true because

$$\|A \cdot DT^{i-j}(z_j)e_{i-j}\| < \frac{|B|}{\angle(e, w_j^*)} b^{i-j} < |B| b^{\frac{i-j}{2}} < b^{\frac{i-j}{2}} \|B \cdot DT^{i-j}(z_j)u\|.$$

□

## B.7 Distortion during bound periods (Sect. 4.3)

**Sublemma B.5**

$$\sum_{i=1}^{\mu} K \frac{\Delta_i}{d_{\mathcal{C}}(z_i)} << 1.$$

**Proof:** Since $|\xi_s - z_s| < e^{-\beta s}$ for all $s < \mu$, we have $\Delta_i < 2e^{-\beta i}$. Let $h_0 = \frac{1}{2\|DT\|} |\log \delta|$. Then

$$\sum_{i=h_0+1}^{\mu} K \frac{\Delta_i}{d_{\mathcal{C}}(z_i)} < K \sum_{i=h_0+1}^{\infty} K e^{-(\beta-\alpha)i} < K \frac{e^{-(\beta-\alpha)h_0}}{1 - e^{-(\beta-\alpha)}} << 1$$

provided $\delta$ is sufficiently small. Also,

$$\sum_{i=1}^{h_0} K \frac{\Delta_i}{d_{\mathcal{C}}(z_i)} < (\sum_{i=1}^{h_0} K (e^{\alpha}\|DT\|)^i)\delta << 1$$

by our choice of $h_0$. □

**Proof of Lemma 4.9**: (cf. [BC2], Lemma 7.8) Assuming the lemma for all $i < \mu$, we give the proof of (8) for step $\mu$; the bound in (9) is proved similarly. For notational simplicity, we drop the hat in $\hat{w}_\mu^*(\xi_0)$.

**Case 1** No fold period expires at $z_\mu$ and $\mu-1$ is not a return time. In this case $w_\mu^*(\cdot) = DT(\cdot)w_{\mu-1}^*(\cdot)$. Writing $C = DT(z_{\mu-1})$, $C' = DT(\xi_{\mu-1})$,

$$u = \frac{w_{\mu-1}^*(z_0)}{\|w_{\mu-1}^*(z_0)\|} \quad \textit{and} \quad u' = \frac{w_{\mu-1}^*(\xi_0)}{\|w_{\mu-1}^*(\xi_0)\|},$$

we have

$$\begin{aligned} \frac{M'_\mu}{M_\mu} &= \frac{M'_{\mu-1}}{M_{\mu-1}} \cdot \frac{\|C'u'\|}{\|Cu\|} \le \frac{M'_{\mu-1}}{M_{\mu-1}} \left(1 + \frac{\|C'u' - Cu\|}{\|Cu\|}\right) \\ &\le \frac{M'_{\mu-1}}{M_{\mu-1}} \left(1 + \frac{\|C' - C\|}{\|Cu\|} + \frac{\|C(u-u')\|}{\|Cu\|}\right). \end{aligned}$$

Since $\|Cu\| > K^{-1}\delta$, $\|C - C'\| < K|\xi_{i-1} - z_{i-1}|$ and $\|u - u'\| \sim |\theta'_{\mu-1} - \theta_{\mu-1}| < Kb^{\frac{1}{2}}\Delta_{\mu-2}$, we have

$$\frac{M'_\mu}{M_\mu} \le \frac{M'_{\mu-1}}{M_{\mu-1}} \cdot \left(1 + K\frac{\Delta_{\mu-1}}{d_{\mathcal{C}}(z_{\mu-1})}\right).$$

**Case 2** $\mu - 1$ is a return time. Then

$$w^*_{\mu-1}(z_0) = A(z_{\mu-1}) \cdot e(z_{\mu-1}) + B(z_{\mu-1}) \cdot w_0.$$

Let

$$A_0 = \frac{A(z_{\mu-1})}{\|w^*_{\mu-1}(z_0)\|}; \qquad B_0 = \frac{B(z_{\mu-1})}{\|w^*_{\mu-1}(z_0)\|}.$$

Then since $w^*_\mu(z_0) = B(z_{\mu-1}) \cdot DT(z_{\mu-1})w_0$, we have

$$\frac{M'_\mu}{M_\mu} = \frac{M'_{\mu-1}}{M_{\mu-1}} \cdot \frac{|B'_0|}{|B_0|} \cdot \frac{\|C'w_0\|}{\|Cw_0\|}.$$

Also with $|B_0| \sim d_{\mathcal{C}}(z_{\mu-1})$ and $|B'_0 - B_0| \le |\theta'_{\mu-1} - \theta_{\mu-1}| + \|e - e'\|$, we get

$$\left|\frac{B'_0}{B_0} - 1\right| < K\frac{\Delta_{\mu-1}}{d_{\mathcal{C}}(z_{\mu-1})}. \tag{32}$$

For the last ratio,

$$\frac{\|C'w_0\|}{\|Cw_0\|} \le 1 + K|\xi_{\mu-1} - z_{\mu-1}|.$$

This finishes the computation for the magnitude. We record also the estimate

$$|A_0 - A'_0| < K\Delta_{\mu-1}$$

for use in Case 3.

**Case 3** There exists a return time $j$ whose fold period expires at time $\mu$. In this case

$$w^*_\mu(z_0) = B(z_j) \cdot DT^{\mu-j}(z_j)w_0 + A(z_j) \cdot DT^{\mu-j}(z_j)e(z_j).$$

Let

$$\begin{aligned} B_0 &= \frac{B(z_j)}{\|w^*_j(z_0)\|}\ , \qquad & A_0 &= \frac{A(z_j)}{\|w^*_j(z_0)\|}\ , \\ C &= DT^{\mu-j}(z_j)w_0\ , \qquad & Y &= DT^{\mu-j}(z_j)e(z_j)\ . \end{aligned}$$

As before, all the corresponding quantities for $\xi_0$ carry a prime. Then

$$\frac{M'_\mu}{M_\mu} = \frac{M'_j}{M_j} \cdot \frac{\|B'_0C' + A'_0Y'\|}{\|B_0C + A_0Y\|} \le \frac{M'_j}{M_j} \cdot \frac{\|C'\|}{\|C\|} \cdot \frac{|B'_0|}{|B_0|} \cdot \left(1 + \frac{\left\|\frac{C'}{\|C'\|} - \frac{C}{\|C\|} + \frac{A'_0Y'}{B'_0\|C'\|} - \frac{A_0Y}{B_0\|C\|}\right\|}{\left\|\frac{C}{\|C\|} + \frac{A_0Y}{B_0\|C\|}\right\|}\right).$$

Since $\frac{|A_0|}{|B_0|} \sim \frac{1}{d_{\mathcal{C}}(z_j)}$ and $\frac{\|Y\|}{\|C\|} \leq d_{\mathcal{C}}^2(z_j)$, it follows that $\frac{\|A_0 Y\|}{\|B_0 C\|} << 1$, giving

$$\frac{M'_\mu}{M_\mu} \leq \frac{M'_j}{M_j} \cdot \frac{\|C'\|}{\|C\|} \cdot \frac{|B'_0|}{|B_0|} \cdot \left(1 + 2\left\|\frac{C'}{\|C'\|} - \frac{C}{\|C\|}\right\| + 2\left\|\frac{A'_0 Y'}{B'_0\|C'\|} + \frac{A_0 Y}{B_0\|C\|}\right\|\right).$$

Since both $\{z_s\}_{s=j}^{\mu}$ and $\{\xi_s\}_{s=j}^{\mu}$ are bound to a critical segment $\{\eta_s\}_{s=0}^{\mu-j}$, $\eta_0 \in \Gamma_{\theta N}$, we have

$$\frac{\|C'\|}{\|C\|} \leq 1 + K \sum_{s=1}^{\mu-j-1} \frac{\hat{\Delta}_s}{d_{\mathcal{C}}(\eta_s)} \leq 1 + K \sum_{s=1}^{\mu-j-1} \frac{\Delta_{s+j}}{d_{\mathcal{C}}(z_{s+j})} = 1 + K \sum_{i=j+1}^{\mu-1} \frac{\Delta_i}{d_{\mathcal{C}}(z_i)}$$

where

$$\hat{\Delta}_s = \sum_{j=1}^{s} (Kb)^{\frac{j}{4}} |z_{s-j} - \xi_{s-j}|. \tag{33}$$

The factor $\frac{|B'_0|}{|B_0|}$ is estimated in (32). This term has no cumulative effect because it is a one-time addition to the exponent in the distortion formula for any given return. Next

$$\left\|\frac{C'}{\|C'\|} - \frac{C}{\|C\|}\right\| < \hat{\theta}$$

where $\hat{\theta}$ is the angle between $C$ and $C'$, which is smaller than $\hat{\Delta}_{\mu-j-1}$. Now

$$\left\|\frac{A'_0 Y'}{B'_0\|C'\|} - \frac{A_0 Y}{B_0\|C\|}\right\| \leq \frac{|A_0|}{|B_0|} \cdot \frac{\|Y' - Y\|}{\|C\|} + \left\|\frac{A_0}{B_0\|C\|} - \frac{A'_0}{B'_0\|C'\|}\right\| \|Y'\|.$$

For the first term we have

$$\frac{|A_0|}{|B_0|} \sim \frac{1}{d_{\mathcal{C}}(z_j)}, \qquad \|Y' - Y\| \leq (Kb)^{\mu-j}|\xi_j - z_j|,$$

and $\|C\| > 1$, where the estimate on $\|Y' - Y\|$ is from (5) in Corollary 2.2. For the second term,

$$\begin{aligned}\left\|\frac{A_0}{B_0\|C\|} - \frac{A'_0}{B'_0\|C'\|}\right\| \|Y'\| &\leq (Kb)^{\mu-j}\frac{|A'_0|}{|B'_0|} \cdot \frac{1}{\|C\|} \cdot \left(\left|\frac{A_0}{A'_0} \cdot \frac{B'_0}{B_0} - 1\right| + \left|1 - \frac{\|C\|}{\|C'\|}\right|\right) \\ &\leq \frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)}\left(\frac{|A_0|}{|A'_0|}\left|\frac{B'_0}{B_0} - 1\right| + \left|\frac{A_0}{A'_0} - 1\right| + \left|1 - \frac{\|C\|}{\|C'\|}\right|\right).\end{aligned}$$

We again estimate term by term: For the first term,

$$\frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)} \cdot \frac{|A_0|}{|A'_0|} \cdot \left|\frac{B'_0}{B_0} - 1\right| \leq \frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)} \cdot \frac{\Delta_j}{d_{\mathcal{C}}(z_j)} < \frac{(Kb)^{\frac{\mu-j}{2}}}{d_{\mathcal{C}}(z_j)} \cdot \Delta_j$$

because $b^{\frac{\mu-j}{2}} < d_{\mathcal{C}}(z_j)$ by the definition of fold period. For the second term,

$$\frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)} \cdot \left|\frac{A_0 - A'_0}{A'_0}\right| \leq \frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)}\Delta_j.$$

Finally, for the third term, we have

$$\frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)}\left|1 - \frac{\|C'\|}{\|C\|}\right| \leq \frac{(Kb)^{\mu-j}}{d_{\mathcal{C}}(z_j)} \sum_{s=1}^{\mu-j-1} \frac{\hat{\Delta}_s}{d_{\mathcal{C}}(z_{s+j})}$$

where $\hat{\Delta}_s$ is as in (33). We also have $b^{\frac{\mu-j}{2}} < d_{\mathcal{C}}(z_{s+j})$, for no fold period starting at time $s+j$ extends beyond index $\mu$. Also, $b^{\frac{\mu-j}{2}} \cdot \hat{\Delta}_s \leq \Delta_j$ for all $s$, $0 < s < \mu - j$. Therefore the third term is again bounded by $K\frac{\Delta_j}{d_{\mathcal{C}}(z_j)}$.

Observe further that if we replace $z_0$ by another point $\xi'_0$ which is bounded to $z_0$, the same argument above continues to work with $\Delta_i(\xi_0, z_0)$ replaced by $\Delta_i(\xi_0, \xi'_0)$. This completes the proof. $\square$

## B.8 Quadratic behavior (Sect. 4.3)

Let $\xi_0(s)$ and $z_0$ be as in Lemma 4.11. We begin with the following

**A priori estimate on $\xi_\mu(s) - z_\mu$ :** (cf. [BC2], p.144-147) Let $t_0(s)$ be a unit vector to $\gamma$ at $\xi_0(s)$, and let $t_\mu = DT^\mu t_0$. We split $t_0$ using $e_\mu$ to get

$$t_0 = A_0 e_\mu + B_0 \binom{0}{1}, \quad \text{so that} \quad t_\mu = A_0 DT^\mu e_\mu + B_0 w_\mu.$$

Writing

$$w_\mu = w_\mu(0) + (w_\mu - w_\mu(0)) = w_\mu(0) + (w^*_\mu - w^*_\mu(0)) + (E_\mu - E_\mu(0))$$

where

$$E_\mu = \sum_{j \in S_\mu} A_j DT^{\mu - j} e_{\ell_j}$$

and $S_\mu$ is the collection of $j$ such that the fold period begun at time $j$ extends beyond time $\mu$, we have

$$\xi_\mu(s) - z_\mu \;=\; \int_0^s t_\mu(u) du \;=\; w_\mu(0) \int_0^s B_0(u) du \;+\; I \;+\; II \;+\; III \tag{34}$$

where

$$I = \int_0^s A_0 DT^\mu e_\mu, \quad II = \int_0^s B_0(w^*_\mu - w^*_\mu(0)), \quad III = \int_0^s B_0(E_\mu - E_\mu(0)).$$

Since $A_0 \approx 1$, $\|I\| \le (Kb)^\mu s$. We claim that

$$\|II\|,\ \|III\| \;\le K e^{2\alpha\mu} \|w^*_\mu(0)\| \int_0^s u\ (\sup_{i \le \mu} |z_i - \xi_i(u)|)\ du.$$

The norm of $II$ is estimated using the distortion estimate in Lemma 4.9. To estimate $\|III\|$, we have, for each $j \in S_\mu$,

$$\begin{aligned} &\|A_j DT^{\mu-j}(\xi_j) e(\xi_j) - A_j(0) DT^{\mu-j}(z_j) e(z_j)\| \\ &\le (Kb)^{\mu-j} |A_j - A_j(0)| + |A_j(0)|\ \|DT^{\mu-j}(\xi_j) e(\xi_j) - DT^{\mu-j}(z_j) e(z_j)\|. \end{aligned}$$

From the distortion estimate in appendix B.7,

$$|A_0 - A_0(0)| < K \|w^*_j(0)\| e^{2\alpha j} \sup_{i \le j} |z_i - \xi_i|.$$

For the second term we have $|A_j(0)| \le \|w^*_j(0)\| e^{\alpha j}$ because $w^*_j(0)$ splits correctly at time $j$, and $\|w^*_j(0)\| \le e^{\alpha\mu} \|w^*_\mu(0)\|$ by Lemma 4.7. Finally, $\|DT^{\mu-j}(\xi_j)e(\xi_j) - DT^{\mu-j}(z_j)e(z_j)\| \le (Kb)^{\mu-j}|\xi_j - z_j|$ by Corollary 2.2, and $B_0(s) \approx 2K_1 s$.

**Proof of Lemma 4.11:** We will show that for the $\mu$ and $s$ in question, the first term in (34) is the dominating one. Let

$$U_\mu \;:=\; K e^{4\alpha\mu} \sup_{j \le \mu} \|w^*_j(0)\|$$

where $K$ is the constant in the bound for $\|II\|$ and $\|III\|$ above. We choose $\mu_0$ large enough that $e^{7\alpha\mu_0} e^{-\beta\mu_0} << 1$, and assume $\delta$ is small enough that $U_{\mu_0}\delta^2 << 1$. We will show inductively first the weaker statement

(i) $|\xi_j(s) - z_j| < U_j s^2$

and then the stronger statement

(ii) $|\xi_j(s) - z_j| = K_1(1 \pm \varepsilon_1) \|w_j(0)\| s^2$.

Assume this has been done for all $j < \mu$. To prove (i) for $j = \mu$, we need to first verify that $U_\mu s^2 << 1$. In the case where $\mu > \mu_0$, we use

$$\sup_{j \le \mu} \|w_j^*(0)\| < e^{\alpha\mu}\|w_\mu^*(0)\| \le e^{2\alpha\mu}\|w_\mu(0)\| \le Ke^{2\alpha\mu}\|w_{\mu-1}(0)\|$$

(see Lemmas 4.7 and 4.6) combined with (ii) for step $\mu - 1$ to get

$$U_\mu s^2 \le Ke^{4\alpha\mu}\ (Ke^{2\alpha\mu}\|w_{\mu-1}(0)\|)s^2 \approx K^2 e^{6\alpha\mu} K_1^{-1}|\xi_{\mu-1}(s) - z_{\mu-1}|$$

$$< K^2 e^{6\alpha\mu} K_1^{-1} e^{-\beta(\mu-1)} << 1.$$

Noting that $\int_0^s B_0 \approx K_1 s^2$, we see from our *a priori* estimate that

$$|\xi_\mu(s) - z_\mu| \ \le\ \|w_\mu(0)\|K_1 s^2 \ +\ (Kb)^\mu s \ +\ 2U_\mu \int_0^s Ku\ [\sup_{i<\mu} |z_i - \xi_i(u)|]\ du.$$

With the quantity inside square brackets being $< U_\mu u^2$ by (i) from the previous step, this is

$$< (U_\mu s^2)e^{-\alpha\mu} + (Kb)^{\frac{\mu}{2}} s^2 + K(U_\mu s^2)^2 \ <\ U_\mu s^2.$$

The proof of (ii) for step $\mu$ now follows immediately. □

## B.9 Proof of Lemma 6.2 (Sect. 6.2)

We begin with a scenario for which one sees easily that the assertion in this lemma holds: Suppose for $1 \le j \le i - s$, $\|DT^j(z_s)\| > \kappa^j$ for some $\kappa >> b^{\frac{1}{2}}$, and that $z_s$ is bounded away from $\mathcal{C}^{(0)}$. Then $e_{i-s}(z_s)$ is well defined and has slope $> K^{-1}$. Suppose, in addition, that $z_s$ is out of all fold periods, so that $w_s$ is a $b$-horizontal vector. Then

$$\|DT^{i-s}(z_s)\|\ \|w_s\| \le K\|DT^{i-s}(z_s)w_s\| = \|w_i\|.$$

This together with $\|w_s\| > c^{c''s}$ (which follows from $\|w_s^*\| > e^{cs}$) gives the desired estimate.

Now, intuitively, the behavior of $\|DT^j(z_s)\|$ is a little different just before or after a return to $\mathcal{C}^{(0)}$. This motivates the following definition: If $t$ is a return time to $\mathcal{C}^{(0)}$ for $z_0$, let $\ell_t$ denote its fold period and let $I_t := (t - 5\ell_t, t + \ell_t)$.

**Claim B.1** *By modifying $I_t$ slightly to $\tilde{I}_t = (t - (5 \pm \varepsilon)\ell_t,\ t + (1 \pm \varepsilon)\ell_t)$, we may assume they have a nested structure.*

*Proof of Claim B.1:* We consider $t = 0, 1, 2, \cdots$ in this order, and determine, if $t$ is a return time, what $\tilde{I}_t$ will be. The right end point of $\tilde{I}_t$ is determined by the following algorithm: Go to $t + \ell_t$, and look for the largest $t'$ *inside the bound period initiated at time $t$* with the property that $t' - 5\ell_{t'} < t + \ell_t$. If no such $t'$ exists, then $t + \ell_t$ is the right end point of $\tilde{I}_t$. If $t'$ exists, then the new candidate end point is $t' + \ell_{t'}$, and the search continues. For the same reasons as in Sect. 4.1, the increments in length are exponentially small and the process terminates.

As for the left end point of $\tilde{I}_t$, it is possible that $t - 5\ell_t \in \tilde{I}_{t'}$ for some $t'$ the bound period initiated at which time does not extend to time $t$. This means that $\ell_{t'} << \ell_t$, and since we assume a nested structure has been arranged for $\tilde{I}_{t'}$ for all $t' < t$, we simply extend the left end of $\tilde{I}_t$ to include the largest $\tilde{I}_{t'}$ that it meets. ♢

Let us assume this nested structure and write $I_t$ instead of $\tilde{I}_t$ from here on.

**Claim B.2** *For $s \notin \cup I_t$, we have, for all $j$ with $1 \leq j < i - s$,*

$$\|w_{s+j}\| \geq b^{\frac{j}{9}} \|w_s\|.$$

*Proof of Claim B.2:* We fix $j$ and let $r$ be such that $z_r$ makes the deepest return between times $s$ and $s+j$. Let $j'$ be the smallest integer $\geq j$ such that $z_{s+j'}$ is outside of all fold periods. Then from Sect. 4.2 , it follows that

$$\|w_{s+j}\| \geq K^{-K\theta(j'-j)} \|w_{s+j'}\| \geq K^{-K\theta(j'-j)} d_{\mathcal{C}}(z_r) \|w_s\| \approx K^{-K\theta(j'-j)} b^{\frac{\ell_r}{2}} \|w_s\|. \tag{35}$$

*Case 1.* $s + j \notin I_r$. In this case, $6\ell_r < j$ since $I_r$ is sandwiched between $s$ and $s + j$, and $j' - j \leq \ell_r$ because $r$ is the deepest return. The rightmost quantity in (35) is therefore $> K^{-\ell_r} b^{\frac{\ell_r}{2}} \|w_s\| > b^{\frac{j}{9}} \|w_s\|$.

*Case 2.* $s + j \in I_r$. The argument is as above, except we only have $5\ell_r < j$.
This completes the proof of the claim. ♢

As noted in the first paragraph, Claim B.2 implies the assertion in Lemma 6.2 for $s \notin \cup I_t$ provided $z_s$ is bounded away from $\mathcal{C}^{(0)}$. This last proviso is easily removed by considering $z_{s+1}$ if necessary.

It remains to prove the lemma for $s \in \cup I_t$. Let $I_r$ be the maximal $I_t$-interval containing $s$. Observe that $6\ell_r < K\alpha\theta s$ (recall that $z_0$ obeys (IA2)) and $\|w_i\| > e^{c''i}$ for some $c'' > 0$. If $i \in I_r$, then $\|DT^{i-s}(z_s)\| < K^{6\ell_r} << e^{\frac{1}{2}c''i} < e^{-\frac{1}{2}c''s} e^{c''i} < e^{-\frac{1}{2}c''s} \|w_i\|$. If $i \notin I_r$, let $s' = r + \ell_r$. Then $s' \notin \cup I_t$, and

$$\|DT^{i-s}(z_s)\| \leq \|DT^{s'-s}(z_s)\| \cdot \|DT^{i-s'}(z_{s'})\| \leq K^{6\ell_r} \cdot K e^{-c's'} \|w_i\|.$$

## B.10 Initial data for critical curves (Sect. 6.3)

**Proof of Lemma 6.4:** Let $J_i := [\hat{a} - \rho^{2i}, \hat{a} + \rho^{2i}]$. Assume for all $i < n$ that the following has been proved:

(i) $J_i \subset \tilde{\Delta}_i$;
(ii) $\Gamma_i^i(\hat{a})$ has a smooth continuation on $J_i$ and $\mathcal{C}^{(i)}$ deforms continuously;
(iii) for all $z \in \Gamma_{i,i}$, $\|\frac{dz}{da}\| \leq K^i$.

We now prove (i)–(iii) for $i = n$.

First we verify that for all $a \in J_n$ and $z_0 \in \Gamma_{n-1,n-1}$, (IA2) and (IA4) hold up to time $n$. This is true for $a = \hat{a}$. For $a \in J_n$, $|z_0(a) - z_0(\hat{a})| < \rho^{2n} K^{n-1}$, so that $|z_j(a) - z_j(\hat{a})| < \rho^{2n} K^{2n}$ for all $j \leq n$. We may assume that $\rho K$ is $<< 1$. It then follows from the discussion at the beginning of Sect. 6.3.1 that $\Gamma_{n,n}(a)$ is well defined, proving (i).

To prove (ii), we fix an arbitrary $\tilde{a} \in J_n$, a component $Q^{(n-1)}$ of $\mathcal{C}^{(n-1)}$, and show that every segment of $\partial R_n(\tilde{a}) \cap Q^{(n-1)}(\tilde{a})$ has a continuation to a segment of $\partial R_n(a) \cap Q^{(n-1)}(a)$. Let $\tilde{\omega}$ be a segment of this kind, and let $\omega(a) := T_a^n(2T_{\tilde{a}}^{-n}\tilde{\omega})$ where $2T_{\tilde{a}}^{-n}\tilde{\omega}$ refers to the segment in $\partial R_0$ with the same midpoint as $T_{\tilde{a}}^{-n}\tilde{\omega}$ and two times as long. Observe that as we vary our parameter from $\tilde{a}$ to $a$, the segment $\omega(a)$ cannot intersect the horizontal boundaries of $Q^{(n-1)}(a)$. Thus the only way $\omega(a)$ can fail to traverse fully $Q^{(n-1)}(a)$ is that it has moved sufficiently far from $\omega(\tilde{a})$ in the *horizontal* direction. We know this cannot happen because $|T_{\tilde{a}}^n - T_a^n| \leq \rho^{2n} K^n$ which is $<< \rho^n$. This proves (ii).

It remains to prove (iii). Consider $\bar{z}(a) = (\bar{x}(a), \bar{y}(a)) \in \Gamma_{n,n}(a)$, and let $y = \psi(x, a)$ denote the $C^2(b)$-curve in $\partial R_n(a)$ containing $\bar{z}(a)$. Then

$$q_n(\bar{x}(a), \psi(\bar{x}(a), a), a) = \partial_x \psi(\bar{x}(a), a)$$

where $q_n(x,y,a)$ is the slope of the contractive vector of order $n$ at $z=(x,y)$. Taking derivative with respect to $a$ on both sides of the last equation, we have

$$\partial_x q_n \cdot \frac{d\bar{x}}{da} + \partial_y q_n \cdot (\partial_x \psi \cdot \frac{d\bar{x}}{da} + \partial_a \psi) + \partial_a q_n = \partial_{xx}\psi \cdot \frac{d\bar{x}}{da} + \partial_{ax}\psi.$$

This implies

$$\frac{d\bar{x}}{da} = \frac{\partial_{xa}\psi - \partial_y q_n \cdot \partial_a \psi - \partial_a q_n}{\partial_x q_n + \partial_y q_n \cdot \partial_x \psi - \partial_{xx}\psi}. \tag{36}$$

Since $\partial_x \psi,\ \partial_{xx}\psi = \mathcal{O}(b)$, $|\partial_x q_n| > K_1$ and $|\partial_y q_n| < K$ (see Corollary 2.2 and Lemma 2.9), the denominator on the right-hand side is bounded away from zero. In the numerator, we have $|\partial_y q_n|, |\partial_a q_n| < K$, and we need to estimate $\partial_a \phi(x,a)$ and $\partial_{ax}\phi(x,a)$.

For this purpose we write the horizontal curve $y=\psi(x,a)$ in parametric form $x = X(t,a), y = Y(t,a)$ where $t$ is the x-coordinate of $T_a^{-n}(x,y)$, i.e., $(t, \pm b) \in \partial R_0$ and

$$(X(t,a), Y(t,a)) = T_a^n(t, \pm b).$$

Let $t=t(x,a)$ be defined by $\psi(x,a) = Y(t(x,a),a)$. Then

$$\partial_a \psi = \partial_t Y(t,a) \cdot \partial_a t(x,a) + \partial_a Y(t,a).$$

Clearly, $|\partial_t Y(t,a)| < K^n b$ and $|\partial_a Y(t,a)| < K^n$. One way to bound $\partial_a t(x,a)$ is to write it as

$$\partial_a t(x,a) = -\frac{\partial_a X(t,a)}{\partial_t X(t,a)}.$$

Since $|\partial_t X(t,a)| > 1$ (recall that $T_a^n|\partial R_0$ is controlled), this term is also $< K^n$. Similar considerations yield $|\partial_{ax}\psi(x,a)| < K^n$. We have proved $\frac{d\bar{x}}{da} < K^n$. The corresponding estimate for $\frac{d\bar{y}}{da}$ follows immediately since

$$\frac{d\bar{y}}{da} = \partial_x \psi \frac{d\bar{x}}{da} + \partial_a \psi.$$

We record an estimate needed in the proof of Lemmas 6.5 and 6.6. Taking derivatives with respect to $a$ one more time on both sides of (36) and estimating corresponding terms (using again Corollary 2.2 and Lemma 2.9), we obtain $|\frac{d^2\bar{x}}{da^2}| < K^n$. This estimate requires that $T_{a,b}$ be $C^3$.

Recall the following lemma due to Hadamard:

**Lemma B.1** (Hadamard) *Let $g \in C^2(0,L)$ be such that $|g| \le M_0$ and $|g''| < M_2$. If $4M_0 < L^2$, then*

$$|g'| \le \sqrt{M_0}(1+M_2).$$

**Proof of Lemma 6.5:** Let $z^{(n)} = (x^{(n)}, y^{(n)})$. For our pusposes, let $g(a) = x^{(n)}(a) - x^{(n-1)}(a)$ and $L = 2\rho^{2n}$. Then $M_0 = b^{\frac{n}{4}}$ and $M_2 = K^n$. Thus $|\frac{dx^{(n)}}{da}| < b^{\frac{n}{8}}K^n < b^{\frac{n}{9}}$. A similar estimate holds for $y^{(n)}$. □

**Proof of Lemma 6.6:** Let $z^n(a) = (x^n(a), y^n(a))$, $z^m(a) = (x^m(a), y^m(a))$, and let $y=\psi(x,a)$ be the $C^2(b)$-curve segment in $\partial R_n$ containing both $z^n(a)$ and $z^m(a)$. Arguments similar to those used to prove $|\partial_{xa}\psi| < K^n$ can also be used to prove that the $C^3$-norm of $\psi$ is $< K^n$.

Let $P_n$ and $Q_n$ be the numerator and denominator on the right hand side of (36), and similarly for $P_m$ and $Q_m$. Then

$$\frac{dx^n}{da} - \frac{dx^m}{da} = \frac{P_n Q_m - P_m Q_n}{Q_n Q_m} = \frac{(P_n - P_m)Q_n + (Q_m - Q_n)P_n}{Q_n Q_m}.$$

As observed in the proof of Lemma 6.4, $|Q_m|, |Q_n| > K^{-1}$. $|Q_m|, |P_n| < K^n$. It remains therefore to estimate $|Q_m - Q_n|$ and $|P_m - P_n|$. Let $q_n$ and $q_m$ denote the slopes of $e_n$ and $e_m$ respectively. Fixing $a$ and omitting it in the arguments of the functions below, we have

$$\begin{aligned} |Q_m - Q_n| &\le |\partial_x q_m(z^m) - \partial_x q_n(z^n)| + |\partial_y q_m(z^m) \cdot \partial_x \psi(z^m) - \partial_y q_n(z^n) \cdot \partial_x \psi(z^n)| \\ &\quad + |\partial_{xx}\psi(z^m) - \partial_{xx}\psi(z^n)|. \end{aligned}$$

The second difference, for example, is

$$\begin{aligned} &\le |\partial_y q_n(z^n)|\, |\partial_x \psi(z^m) - \partial_x \psi(z^n)| + |\partial_x \psi(z^m)|\, |\partial_y q_m(z^m) - \partial_y q_n(z^m)| \\ &\quad + |\partial_x \psi(z^m)|\, |\partial_y q_n(z^m) - \partial_y q_n(z^n)|. \end{aligned}$$

This is $< (Kb)^{\frac{n}{4}}$ since $\|\psi\|_{C^3} < K^n$, $|\partial_{xx} q|, |\partial_x \partial_y q| < K$ (Corollary 2.2), $|q_m(z^n) - q_n(z^n)| < (Kb)^n$ (Lemma 2.1) and $|z^m - z^n| < (Kb)^{\frac{n}{4}}$ (Lemma 2.10). The other terms in $|Q_m - Q_n|$ and $|P_m - P_n|$ are estimated similarly. □

## B.11 Dynamics of critical curves (Sect. 6.4)

**Proof of Lemma 6.8:** Let $\hat{z}_0$ be an arbitrary critical point. First we observe that as functions of $a$, $z_i(a)$ and $\hat{z}_0(a)$ move at very different speeds: $\|\frac{d}{da} z_i(a)\| \sim \|w_i(a)\| > e^{ci}$ by Proposition 6.1, whereas from Sect. 6.3 we have $\|\frac{d}{da}\hat{z}_0(a)\| < K$.

Next we consider $z_i(a) \in Q^{(k-1)}(a) \setminus Q^{(k)}(a)$ for some $k << i$, so that $\phi_a(z_i(a)) \in \partial Q^{(k-1)}(a)$, and study the relative movements of $z_i$, $\phi(z_i)$ and the relevant critical regions as $a$ varies. For definiteness, let us assume that $z_i$ is in the right component of $(Q^{(k-1)} \cap R_k) \setminus Q^{(k)}$ (which we call $A$), and that it moves left as $a$ increases. (See Fig. 1 in Sect. 1.2.) In horizontal distance, it follows from the first paragraph that *relative to* $\phi(z_i)$, $z_i$ is moving left at a speed $> K^{-1}e^{ci} - K$, which we assume to be $>> 1$. We do not have analytic estimates on the relative vertical movements of $\phi(z_i)$ and $z_i$, but note that since $z_i \notin \partial R_k$, it must enter $A$ through its right vertical boundary and exit through the left. As $z_i$ meets these vertical boundaries, it crosses them instantaneously due again to the horizontal speed differential between $z_i$ and the critical points which determine these regions.

What we have shown is that the function $a \mapsto \phi_a(z_i(a))$ is continuous except at a discrete set of points corresponding to when $z_i(a)$ crosses a vertical boundary of some $Q^{(k)}$. If $a$ and $a'$ are the entry and exit parameters for $Q^{(k-1)} \setminus Q^{(k)}$ as above, we have $|a - a'| < \rho^{k-1}(K^{-1}e^{ci} - K)^{-1}$, and consequently $|\phi_a(z_i) - \phi_{a'}(z_i)| < K'\rho^{k-1}e^{-ci}$. As $z_i$ crosses the vertical boundary into $Q^{(k)}$, a jump in $\phi_a(\cdot)$ occurs due to our rule for selecting binding points; this jump is $< b^{\frac{k-1}{4}}$.

As we continue to move toward the cricial set, either $\gamma_i$ ends or we enter the "last" $Q^{(k)}$ available at step $i$, with $k \sim \theta i$. Let $\bar{a}$ be the parameter that corresponds to the end point of $\gamma_i$ or where $d_{\mathcal{C}(\bar{a})}(z_i(\bar{a})) = e^{-\frac{\alpha i}{2}}$, whichever is reached first, and let $\bar{z} = \phi_{\bar{a}}(z_i(\bar{a}))$. We will use $\bar{z}$ as our "binding point" for $\gamma_i$. The "error" in this choice for $z_i(a)$, i.e. $|\, |z_i(a) - \bar{z}|_h - d_{\mathcal{C}(a)}(z_i(a))\, |$, is less than the total variation of $a \mapsto \phi_a(z_i(a))$ between $a$ and $\bar{a}$. We have proved that this is $< Ke^{-ci} d_{\mathcal{C}(a)}(z_i(a)) + Kb^{\frac{k-1}{4}}$ where $\rho^k \sim d_{\mathcal{C}(a)}(z_i(a))$. □

**Proof of Lemma 6.9:** Let $\tilde{p} = \min\,\{p_a(z_i(a)) : z_i(a) \in I_{\mu j}\}$. Then by Corollary 4.2(a) and the last lemma, $\tilde{p} < K|\mu|$. Assertion (a) in Lemma 6.9 is obvious for $j \le \ell$ where $\ell$ is the common fold period. For $\ell < j \le \tilde{p}$, we have:

$$|z_{i+j}(a) - z_{i+j}(a')| \le \text{length}(\omega_j) = \int_{\omega_0} \frac{\|\tau_{i+j}\|}{\|\tau_i\|} \sim \int_{\omega_0} \frac{\|w_{i+j}\|}{\|w_i\|}.$$

Since $z_i$ is outside of fold periods and $w_i$ splits correctly, we have, for $j > \ell$, $\|w_{i+j}\|/\|w_i\| \sim e^{-\mu}\|w_j(z_i)\|$. Furthermore, if $\hat{z}_0 = \phi(z_i)$ and $\tilde{p} = p(z_i(\tilde{a}))$, then

$$e^{-\mu}\|w_j(z_i)\| \sim e^{-\mu}\|w_j(\hat{z}_0)\| \sim e^{-\mu}\|w_j(\hat{z}_0(\tilde{a}))\|,$$

the first $\sim$ coming from Lemma 4.9, and the second from the fact that $|\phi_a(z_i(a)) - \phi_{\tilde{a}}(z_i(\tilde{a}))| < e^{-ci} << K^j$ for $j < K|\mu| < K\alpha i$. Thus using the distance formula (10) in Lemma 4.11 for $T_{\tilde{a}}$, we have

$$\int_{\omega_0} \frac{\|\tau_{i+j}\|}{\|\tau_i\|} \sim e^{-2\mu}\frac{1}{\mu^2}\|w_j(\hat{z}_0(\tilde{a}))\| \sim \frac{1}{\mu^2}|z_{i+j}(\tilde{a}) - \hat{z}_j(\tilde{a})| < \frac{1}{\mu^2}e^{-\beta j}.$$

This completes the proof of (a); (c) follows from $\|w_{\tilde{p}}(z_i(a))\| \sim \|w_{\tilde{p}}(z_i(\tilde{a}))\|$ and Proposition 6.1.

It remains to prove (b). From (a) we have that $z_{i+\tilde{p}}$ is out of all fold periods whenever $z_{i+\tilde{p}}(\tilde{a})$ is. To show that the slopes of $\tau_{i+\tilde{p}}$ are $< K(\delta)$, we use Lemma 6.3: the $w_{i+\tilde{p}}$-vectors are b-horizontal, so it suffices to show that $\|w_s\| \leq K\|w_{i+\tilde{p}}\|$ for all $s < i + \tilde{p}$. For $s \geq i$, this is true by comparison with $a = \tilde{a}$; for $s < i$, $\|w_s\| \leq \|w_i\|$ because $z_i$ is a free return. Finally, the small slope of $\omega_{\tilde{p}}$ allows us to reverse the inequalities displayed above to conclude that $|\omega_{\tilde{p}}| \geq \frac{1}{\mu^2}e^{-\beta\tilde{p}}$. $\square$

## B.12 Distortion estimate for critical curves (Sect. 6.4)

Let $J$ be a parameter interval satisfying all the assumptions made in Proposition 6.2, and let $a, a' \in J$. Assume that $z_i(a)$ and $z_i(a')$ are free returns, and that they lie in the same $I_{\mu j}$ with $\mu < \alpha i$. Write $\xi_0(a) = z_i(a)$ and $w_k(\xi_0(a)) = DT_a^k(\xi_0(a))\binom{0}{1}$. Let $\tilde{p} = p(z_i(\tilde{a}))$ be the bound period and $\hat{z}_0(\tilde{a}) = \phi(z_i(\tilde{a}))$ the binding point in the proof of Lemma 6.9. For $k < \tilde{p}$, let $\{w_k^*(\xi_0(a))\}$ be given by the splitting algorithm taken with respect to the orbit segment $\{\hat{z}_k(\tilde{a})\}_{k=0}^{\tilde{p}}$, and write $w_k^*(\xi_0(a)) = M_k e^{i\theta_k(\xi_0(a))}$. The corresponding quantities for $\xi_0(a') = z_i(a')$ are defined analogously.

**Sublemma B.6** *For $k < \tilde{p}$,*

$$\frac{M_k(\xi_0(a'))}{M_k(\xi_0(a))}, \quad \frac{M_k(\xi_0(a))}{M_k(\xi_0(a'))} \leq \exp\{K\sum_{j=1}^{k-1}\frac{\Delta_j(a,a')}{d_{\mathcal{C}}(\hat{z}_j(a))}\}$$

*and*

$$|\theta_k(\xi_0(a)) - \theta_k(\xi_0(a'))| < (Kb)^{\frac{1}{2}}\Delta_{k-1}(a,a')$$

*where*

$$\Delta_j(a,a') = \sum_{s=1}^{j}(Kb)^{\frac{s}{4}}(|\xi_{j-s}(a) - \xi_{j-s}(a')| + |a-a'|).$$

**Proof:** The computation is similar to that in Appendix B.7, modulo the following adaptations to accommodate for the fact that different parameter values are involved in the present situation:

(i) Replace $|DT(\xi) - DT(\xi')| < K|\xi - \xi'|$ by

$$|DT_a(\xi(a)) - DT_{a'}(\xi(a'))| < K(|\xi(a) - \xi(a')| + |a-a'|).$$

(ii) Replace $|e - e'| < K|\xi - \xi'|$ by

$$|e(a) - e(a')| < K(|\xi(a) - \xi(a')| + |a-a'|).$$

(iii) Replace $|Y - Y'| < (Kb)^{\mu-j}|\xi - \xi'|$ by

$$|Y(a) - Y(a')| < (Kb)^{\mu-j}(|\xi(a) - \xi(a')| + |a-a'|).$$

$\square$

Next we prove a version of Sublemma B.6 with $\binom{0}{1}$ replaced by $u_i(a) := \frac{w_i(z_0)(a)}{\|w_i(z_0)(a)\|}$.

**Sublemma B.7**
$$\frac{\|DT_a^{\tilde{p}}(\xi_0(a))u_i(a)\|}{\|DT_{a'}^{\tilde{p}}(\xi_0(a'))u_i(a')\|} < \exp\{K\frac{|\xi_0(a)-\xi_0(a')|}{e^{-\mu}}\}$$

**Proof:** The proof uses the fact that both $u_i(a)$ and $u_i(a')$ split correctly. Writing

$$u_i(a) = A(a)e(a) + B(a)\binom{0}{1},$$

we have

$$DT_a^{\tilde{p}}(\xi_0(a))u_i(a) = A(a)DT_a^{\tilde{p}}(\xi_0(a))e(a) + B(a)w_{\tilde{p}}(\xi_0(a)).$$

The proof is similar to that of Case 3 of Lemma 4.9, and Sublemma B.6 is used to compare $w_p(\xi_0(a))$ and $w_p(\xi_0(a'))$. □

**Proof of Proposition 6.2:** In view of Proposition 6.1, it suffices to show that there exists a constant $K > 0$ such that

$$\frac{1}{K} < \frac{|\ w_n(z_0(a))\ |}{|\ w_n(z_0(a'))\ |} < K.$$

Divide the time interval $(1, n)$ into bound and free period according to Lemma 6.9. As usual we denote free return times as $t_k$, $1 \le k < q$, and the bound period at $t_k$ as $p_{t_k}$. Write

$$\log\frac{\|w_n(z_0(a))\|}{\|w_n(z_0(a'))\|} = \sum_{k<q} S'_k + \sum_{k<q} S''_k$$

where

$$S'_k = \log\frac{\|DT_a^{p_k}(z_{t_k}(a))u_{t_k}(a)\|}{\|DT_{a'}^{p_k}(z_{t_k}(a'))u_{t_k}(a')\|}, \qquad S''_k = \log\frac{\|DT_a^{t_{k+1}-p_k}(z_{t_k+p_k}(a))u_{t_k+p_k}(a)\|}{\|DT_{a'}^{t_{k+1}-p_k}(z_{t_k+p_k}(a'))u_{t_k+p_k}(a')\|}.$$

First we prove that $\sum_{k<q} S''_k < K$. Since $\gamma_j \cap \mathcal{C}^{(0)} = \emptyset$ for $t_k + p_k \le j \le t_{k+1}$, it is straightforward to see using Sublemma B.6 that

$$S''_k < \frac{K}{\delta}\sum_{j=t_k+p_k}^{t_{k+1}}(|z_j(a) - z_j(a')| + |a - a'|).$$

The effect of $|a - a'|$ can be ignored since $|a - a'| < e^{-cn}$. By Lemma 6.7, the slopes of $\gamma_j$ are uniformly bounded and the length of $\gamma_j$ grows exponentially, so

$$\sum_{j=t_k+p_k}^{t_{k+1}} |z_j(a) - z_j(a')| < K|\gamma_{t_{k+1}}|.$$

Again by Lemma 6.9(b), $|\gamma_{t_{k+1}}| > K|\gamma_{t_k}|$. Therefore $\sum_{k<q} S''_k < K$.

To estimate $\sum_{k<q} S'_k$ we apply Sublemma B.7. The effect of the term $|a - a'|$ can again be ignored, so that

$$\sum_{k<q} S'_k \le K\sum_{k=1}^{q-1}\frac{\gamma_{t_k}}{e^{-\mu_k}}$$

where $\gamma_{t_k} \in I_{\mu_k j_k}$. To estimate this sum, let $m(\mu) = max\{t_k : \mu_k = \mu\}$ for each $\mu$. Using the fact that $|\ \gamma_{t_{k+1}}\ | \ge K|\gamma_{t_k}|$, we conclude that

$$\sum_{k<q} S'_k < K\sum_{k<q}\frac{|\ \gamma_{t_k}\ |}{e^{-\mu_k}} < K\sum_{\mu}\frac{|\ \gamma_{m(\mu)}\ |}{e^{-\mu}} < K\sum_{\mu}\frac{1}{\mu^2}.$$

This completes the proof. □

# References

[Bal] V. Baladi, *Dynamical zeta functions*, Real and Complex Dynamical Systems (B. Branner and P. Hjorth, ed.), Kluwer Academic Publisher, 1995a.

[Bar] M. Barge, *Prime end rotation numbers associated with the Hénon maps*, Continuum Theory and Dynamical Systems (T. West, ed.,), Lecture Notes in Pure and Applied Math. **149** (1993), Dekker, New York, 15-34.

[BC1] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. Math. **122** (1985), 1-25.

[BC2] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133** (1991), 73-169.

[BV] M. Benedicks and M. Viana, *Solutions of the basin problem for certain non-uniformly hyperbolic attractors*, preprint (1999).

[BY1] M. Benedicks and L.-S. Young, *Sinai-Bowen-Ruell measure for certain Hénon maps*, Invent. Math. **112** (1993), 541-576.

[BY2] M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Asterisque (1999).

[Bi] G.D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc., Providence, RI, 1927.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer (Lecture Notes in Math., Vol. **470**), Berlin, 1975.

[BSC1] L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov, *Markov partitions for two-dimensional hyperbolic billiards*, Russ. Math. Survey **45** (1990), 105-152.

[BSC2] L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov, *Statistical properties of two-dimensional hyperbolic billiards*, Russ. Math. Survey **46** (1991), 47-106.

[CE] P. Collet and J.P. Eckmann, *Positive Liapunov exponents and absolute continuity for maps of the interval*, Ergodic Theory and Dynamical Systems **12** (1992), 13-37.

[CL] P. Collet and Y. Levy, *Ergodic properties of the Lozi mappings*, Comm. Math. Phys. **93** (1984), 461-481.

[C] P. Cvitanović, *Periodic orbits as the skeleton of classical and quantum chaos*, Physica D **51** (1991), 138-151.

[DRV] L.J. Diaz, J. Rocha and M. Viana, *Strange attractors in saddle-node cycles: prevalence and globality*, Invent. Math. **125** 37-74.

[G] J. Guckenheimer, *A strange strange attractor*, Hopf Bifurcation and its Application (J.E. Marsden and M. McCracken, ed.), Springer-Verlag: New York, Heidelberg, Berlin, 368-381.

[GH] J. Guckenheimer and P. Holmes, *Nonlinear oscillators, dynamical systems and bifurcations of vector fields*, Springer-Verlag, Appl. Math. Sciences **42** (1983).

[H] M. Hénon, *A two-dimensional mapping with a strange attractor*, Comm. Math. Phys. **50** (1976), 69-77.

[I1] Y. Ishii, *Towards a kneading theory for Lozi mappings I: A solution of the pruning front conjecture and the first tangency problem*, Nonlinearity **10** (1997), 731-747.

[I2] Y. Ishii, *Towards a kneading theory for Lozi mappings II: Monotonicity of the topological entropy and Hausdorff dimension of attractors*, Commun. Math. Phys. **190** (1997), 375-394.

[J] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39-88.

[Led] F. Ledrappier, *Propriétés ergodiques des mesures de Sinai*, Publ. Math., Inst. Hautes Etud. Sci. **59** (1984), 163-188.

[Lev] M. Levi, *Qualitative analysis of periodically forced relaxation oscillations*, Mem. AMS, **214** (1981), 1-147.

[K] A. Katok, *Lyapunov exponents, entropy, and periodic orbits for diffeomorphisms*, Publ. math. IHES, **51** (1980), 137-173.

[dMvS] W. de Melo and S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, (1993).

[MS] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math., **67**, (1980), 45-67.

[M1] M. Misiurewicz, *Absolutely continuous invariant measures for certain maps of an interval*, Publ. Math. IHES. **53** (1981), 17-51.

[M2] M. Misiurewicz, *The Lozi mapping has a strange attractor*, Nonlinear Dynamics, (R.H.G. Helleman, ed.), New York Academy of Sciences: New York, 1980, 348-358.

[Mil] J. Milnor, *On the concept of attractor*, Comm. Math. Phys., **99** (1985), 177-195.

[MT] J. Milnor and W. Thurston, *On iterated maps of the interval I and II*, 1977 preprint. Published in "Dynamical Systems: Proc. Univ. of Maryland 1986-87", (1988), Lect. Notes in Math., vol. 1342. Springer, Berlin New York, pp. 465-563.

[MV] L. Mora and M. Viana, *Abundance of strange attractors*, Acta. Math. **171** (1993), 1-71.

[NS] T. Nowicki and S. Van Strien, *Absolutely continuous invariant measures under a summability condition.* Invent. Math. **105** (1991), 123-136.

[P] Ja. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russ. math. Surv. **32.4** (1977), 55-114.

[PP] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Société Mathématique de France (Astérisque, vol. **187-188**), Paris, 1990.

[PS] C. Pugh and M. Shub, *Ergodic attractors*, Trans. A. M. S. **312** (1989) 1-54.

[PT] J. Palis and F. Takens, *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge studies in advanced mathematics, **35** Cambridge University Press, Cambridge, (1993).

[R1] D. Ruelle, *A measure associated with Axiom A attractors* Amer. J. Math. **98** (1976), 619-654.

[R2] D. Ruelle, *Thermodynamic Formalism*, Addison Wesley, Reading, MA, 1978.

[R3] D. Ruelle, *Dynamical Zeta functions for piecewise monotone maps of interval*, CRM Monograph Series, Vol. **4** Amer. Math. Soc., Providence, NJ, 1994.

[R4] D. Ruelle, *Ergodic theory of differentiable dynamical systems.* Publ. Math., Inst. Hautes Étud. Sci. **50** (1979), 27-58.

[Ro] C. Robinson, *Homoclinic bifurcation to a transitive attractor of Lorenz type*, Non-linearity **2** (1989), 495-518.

[Ry] M. Rychlik, *Lorenz attractors through Sil'nikov-type bifurcation, Part I*, Ergodic Theory and Dynamical Systems **10** (1990), 793-822.

[S1] Y. G. Sinai, *Dynamical systems with elastic reflections: Ergodic properties of dispersing billiards*, Russ. Math. Surveys **25** (1970), 137-189.

[S2] Y. G. Sinai, *Gibbs measure in ergodic theory*, Russian Math. Surveys **27** (1972), 21-69.

[Sm] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[T] M. Tsujii, *A simple proof for monotonicity of entropy in the quadratic family*, Ergodic Theory and Dynamical Systems, to appear.

[TTY] P. Thieullen, C. Tresser and L.-S. Young, *Positive exponent for generic 1-parameter families of unimodal maps*, C.R. Acad. Sci. Paris, t. 315 Série I (1992), 69-72; *J′Analyse* **64** (1994), 121-172.

[V] M. Viana, *Strange attractors in higher dimensions*, Bull. Braz. Math. Soc. **24** (1993), 13-62.

[W1] R. Williams, *Classification of one-dimensional attractors*, Global Analysis, Proc. Sym. in Pure Math., vol **14** (1970), 341-361.

[W2] R. Williams, *The structure of Lorenz attractors*, Turbulence Seminar Berkeley 1996/97, (P. Bernard and T. Ratiu, ed.), 1977, Springer-Verlag: New York, Heidelberg, Berlin, 1977, 94-112.

[Wa] P. Walters, *An introduction to ergodic theory*, Grad. Texts in Math., Springer-Verlag (1981).

[Y1] L.-S. Young, *A Bowen-Ruelle measure for certain piecewise hyperbolic maps*, Trans. A.M.S. **287** (1985), 41-48.

[Y2] L.-S. Young, *Ergodic theory of differentiable dynamical systems*, Real and Complex Dynamical Systems, (B. Branner and P. Hjorth, ed.), Kluwer Acad. Press, 1995.

[Y3] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. **147** (1998), 585-650.

[Y4] L.-S. Young, *Recurrence time and rate of mixing*, Israel J. of Math. (1999).